≪Next time: problem ⋅ ... ⟩

6.896
3/8/04
L9.1

Optimal retiming

Recall: $W(u,v) = \min\{w(p): u \xrightarrow{p} v\}$
   $D(u,v) = \max\{d(p): u \xrightarrow{p} v$ is a critical path$\}$

Lemma 1. $\Phi(G) \leq c$ iff $W(u,v) \geq 1$ whenever $D(u,v) > c$. ☒

Lemma 2. Let r be legal retiming of G.

1. p is a crit path of G iff p is a crit path of G_r.
2. $W_r(u,v) = W(u,v) - r(u) + r(v)$
3. $D_r(u,v) = D(u,v)$. ☒

Lemma 3. $\Phi(G_r) = D(u,v)$ for some $u,v \in V$.

Proof. Let $u \xrightarrow{p} v$ be path in G_r $\ni$ $\Phi(G_r) = d(p)$ and $w_r(p) = 0$ (def of clock period). Thus, $W_r(u,v) = w_r(p) = 0$, and $D_r(u,v) = d(p)$, since no 0-wt path in G_r has larger delay than p. Thus, $\Phi(G_r) = D_r(u,v) = D(u,v)$. ☒

Theorem. Let $r: V \to \mathbb{Z}$. Then, r is a legal retiming of G $\ni$ $\Phi(G_r) \leq c$ iff
1. $r(u) - r(v) \leq w(e)$ $\forall u \xrightarrow{e} v$.
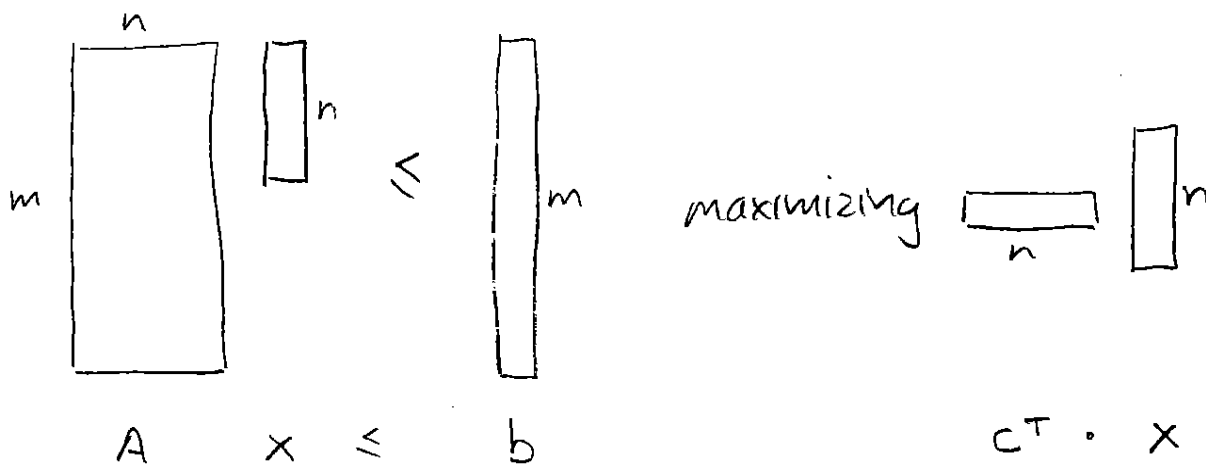2. $r(u) - r(v) \leq W(u,v) - 1$ $\forall u,v \ni D(u,v) > c$.

Proof.
   1. $r(u) - r(v) \leq w(e)$ iff $w_r(e) = w(e) - r(u) + r(v) \geq 0$.
   2. $\Phi(G_r) \leq c$ precisely when $w_r(u,v) \geq 1$ $\forall u,v \in V$ $\ni D_r(u,v) > c$, by Lemma 1. Rewrite using Lemma 2. ☒

$O(E)$ constraints of type 1. $\}$ Linear!
$O(V^2)$ "   "   " 2. $\}$

Linear programming

Let A be an $m \times n$ matrix, b be an m-vector, and c be an n-vector. Find an n-vector x that maximizes $c^T x$ subject to $Ax \le b$, or determine no solution exists.

$$A \quad x \quad \le \quad b \qquad \text{maximizing} \quad c^T \cdot x$$

General algs
- simplex — practical, but w-c exp. time
- interior-pt algs — poly time, becoming practical.

"Feasibility" problem: No opt. criterion.
　　Find x $\ni$ $Ax \le b$.

Difference constraints

Each row of A contains exactly one 1 and one -1, and rest are 0's.

Ex.
$$\left. \begin{array}{l} x_1 - x_2 \le 3 \\ x_2 - x_3 \le -2 \\ x_1 - x_3 \le 2 \end{array} \right\} \quad x_j - x_i \le a_{ij}$$

Solution
$$x_1 = 3$$
$$x_2 = 0$$
$$x_3 = 2$$

Linear prog., but simpler.

Constraint graph

- vertex v_i for each unknown x_i
- edge $v_i \longrightarrow v_j$ with weight a_{ij}
  if $x_j - x_i \leq a_{ij}$ is constraint.

Thm. If constraint graph has neg-wt cycle, then no solution. (Constraints unsatisfiable).

Pf. Sup. cycle is $v_1 \to v_2 \to \cdots \to v_k \to v_1$.

Then,
$$x_2 - x_1 \leq a_{12}$$
$$x_3 - x_2 \leq a_{23}$$
$$\vdots$$
$$x_k - x_{k-1} \leq a_{(k-1)k}$$
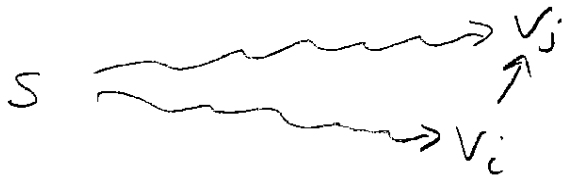$$x_1 - x_k \leq a_{k1}$$

$$\overline{\quad 0 \quad \leq \text{ wt of cycle} < 0 \quad}$$

$\therefore$ No values for x_i satisfy constraints. ☒

Thm. No neg-wt cycle $\Rightarrow$ constraints satisfiable.

Pf. Add new vertex s to V with 0-wt edge to each $v_i \in V$. (No neg-wt cycle introduced).

Let $\delta(s, v_i)$ = wt of sh. path from s to v_i.
(Sh. paths exist, since no neg-wt cycle)
Claim: $x_i = \delta(s, v_i)$ is solution.

$$\delta(s, v_j) \leq \delta(s, v_i) + a_{ij} \qquad (\Delta\text{-ineq.})$$
$$\underset{x_j}{\Vert} \qquad\qquad \underset{x_i}{\Vert} \qquad \Rightarrow x_j - x_i \leq a_{ij} \text{ ☒}$$

Bellman-Ford algorithm

Sh. path from source $s \in V$ to all $v \in V$ or
 determine neg-wt cycle exists.

Init: $d[v] = \begin{cases} 0 & \text{if } v = s \\ \infty & \text{otherwise.} \end{cases}$

for $i \leftarrow 1$ to $|V| - 1$
   do for each edge $(u,v) \in E$
     do $d[v] \leftarrow \min \{d[v], d(u) + w(u,v)\}$.

if $\exists \, (u,v) \; \$ \; d[v] > d[u] + w(u,v)$
   then neg-wt cycle exists.

No neg-wt cycle $\Rightarrow d[v] = \delta(s,v)$.

Correctness: induction (see CLRS)
Running time: $O(VE)$

Opt. clock period

1. Compute W and D — $O(V^3)$
2. Sort elems of D (clock period is one of them) — $O(V^2 \lg V)$
3. Binary search among D elems using Bellman-Ford to test feasibility of LP — $O(V^3 \lg V)$
4. Use values found by B-F to retime.

《 Reminder: problem session next time 》