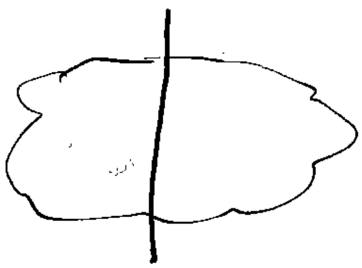
6.896
18.1 4-21-04
BRADLEY C KUSZMAUL

Divide-and-Conquer Layout

~~A general algorithm for layout produces low-area layouts for any bounded degree tree or planar graph,~~
~~+ near-optimal for all~~

A general layout algorithm,
near optimal layout for all graphs
optimal for bounded degree trees + planar graphs

~~Separators~~
~~Idea:~~
Idea:



Layout(G):

(1) Find small number of edges that disconnect the graph into nearly equal pieces. (Can be tough)

(2) Recursively layout the two halves

(3) Put the two halves together, + wire it up.

Separators

~~Defn: G has an~~

Defn: G a graph $G=(V,E)$

$S: \mathbb{Z} \to \mathbb{Z}$

G has an S-separator (i.e. S-separable) if

a) G has 1 vertex, or

b) ~~let g = |V|~~

$\exists$ a set $A \subseteq E$ s.t. $|A| < S(|V|)$

and $(V, E-A)$ is two disconnected graphs

~~G_1, G_2~~ $G_1 = (V_1, E_1)$

$G_2 = (V_2, E_2)$

s.t. $|V_1| \geq |V|/3$ and $|V_2| \geq |V|/3$

[neither G_1 nor G_2 has > twice the nodes of the other]

and G_1 + G_2 are S-separable.

6.896
18.2 4-21-07

~~Example~~:

Thm: Binary trees are 1-separable.

proof: pick a root
travel down the tree looking for a node that is the ancestor of ~~of~~ $n/3$ to $\frac{2n}{3}$ nodes.

~~that node's parent edge disconnects the tree~~

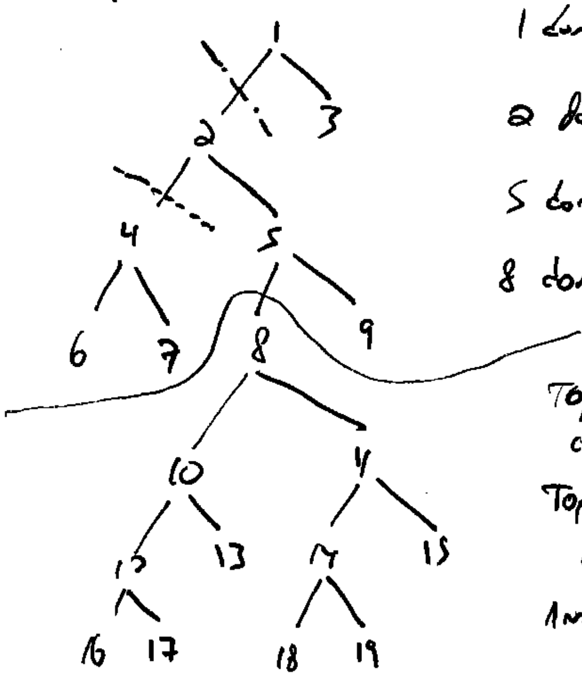
case A: this subtree is like $\frac{n}{3} + \frac{2n}{3}$ nodes
done

case B: ~~subtree too small. we d~~
Subtree too big. $> \frac{2n}{3}$ nodes

One of two children is at least half ~~$\frac{2n}{3}$~~ the nodes,
so go that path

case C: too small. we don't go there.

Example..



1 dominates 19 nodes – too big

2 dominates 17 nodes too big

5 dominates 13 nodes, too big

8 dominates 12 nodes – ok

Top half is 8 nodes
cut here ...... into 3 + 5

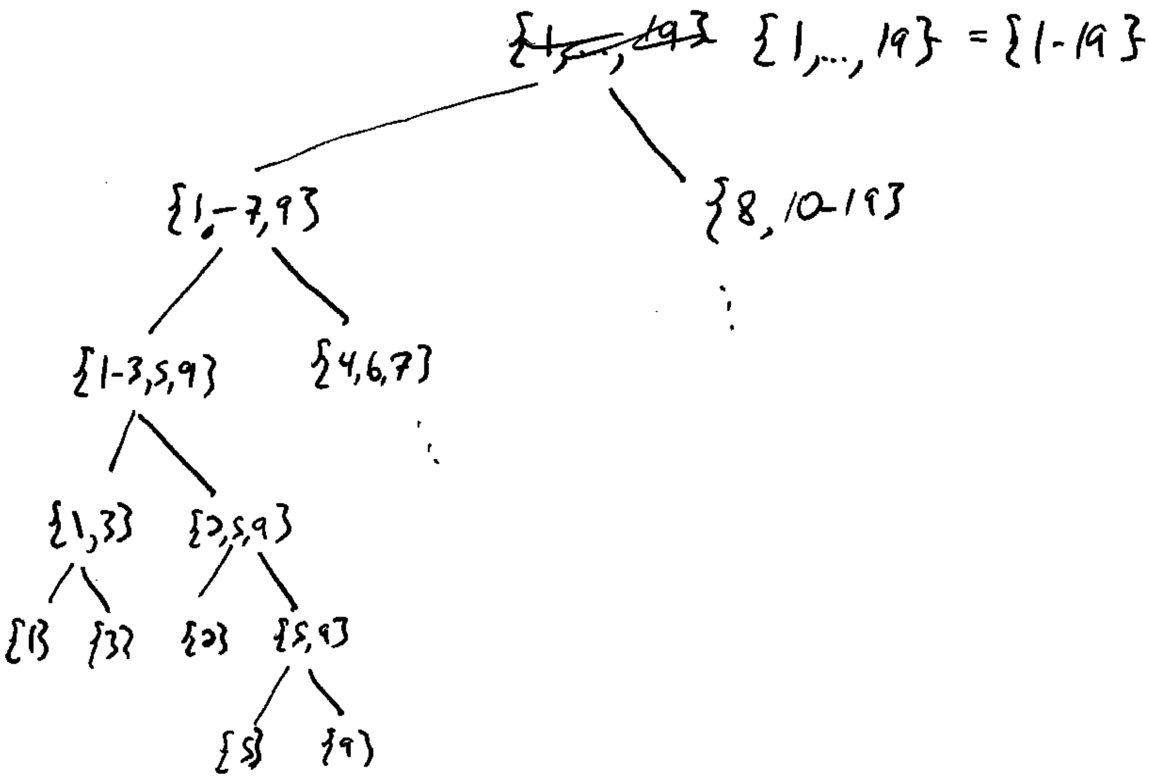
Top 1/4 has 5 nodes
cut here —.—.—.—

And so forth

$\boxed{\text{SAVE THIS TREE}}$

~~6.8.9~~
6.896
18.3 4-21-04

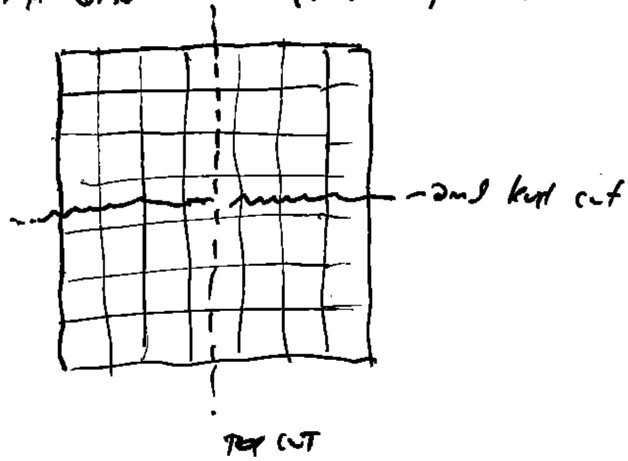
Def'n:
A partition tree of a GRAPH is a ~~tree~~ ~~where each~~
Defined by example:



[its a tree even if G is not a tree]

6.896
18.4
4-21-04

Example: A GRID is $O(\sqrt{n})$ separable



Defn: G has a strong S-separator if the sizes of the subgraphs are at most ~~...~~ $\frac{|V|+1}{2}$.
[cut exactly in half even, otherwise as close as possible]

Example: ~~a good~~ a grid of size $2^i \times 2^i$ is strongly $\sqrt{n}$ separable

Claim: not surprising. we'll show ~~n^ε separable $\Rightarrow$~~
~~S~~ S is $\Omega(n^\varepsilon)$ implies S-separable $\Rightarrow$ strongly separable

Defn: Γ (gamma) defined as

$$\Gamma_S(n) = S(n) + S\left(\tfrac{2}{3}n\right) + S\left(\tfrac{4}{9}n\right) + \cdots$$

$$= \sum_{i=0}^{\lceil \log_{3/2} n \rceil} S\left(\left(\tfrac{2}{3}\right)^i n\right)$$

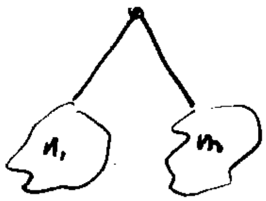
Example: $S(n) = n^\alpha$ then

$$\Gamma_S(n) = n^\alpha + \left(\tfrac{2}{3}n\right)^\alpha + \left(\tfrac{4}{9}n\right)^\alpha + \cdots = n^\alpha \cdot \left(1 + \tfrac{2}{3}^\alpha + \tfrac{4}{9}^\alpha + \cdots\right)$$

$$= n^\alpha \cdot \frac{1}{1 - \left(\frac{2}{3}\right)^\alpha} = O(n^\alpha)$$

6.896
18.6 4-21-04

proof by induction: pick $t < |V|$



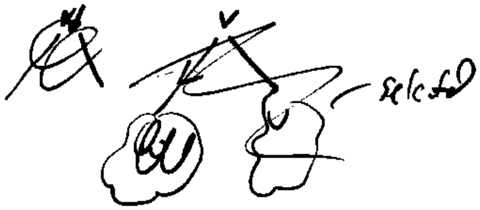
if $n_1 \leq t$ then add the left subtree to our selected set + go right to pick $t - n_1$ elts.

~~if $n_1 + n_2 \geq t$ then pick one side~~

if $n_1 > t$ then go left
don't use the right subtree in
pick n_1 elts from left.

claim: ~~the edges connecting our selected sets to anything else~~
~~Second~~ $\leq \Gamma_S(n)$ edges connecting selected sets to anything else.

pf.



~~If u is selected then at most $S(n)$~~ edges connect u to ~~anything else~~ non-selected set

~~If u is not selected then at most $S(n)$ edges connect u to the other~~
~~v-u to the selected set~~.

~~if v is~~ .

If ~~some~~ u is selected then at most $S(n)$ of the edges to u are needed to connect u to non-selected nodes.

If u is not selected, then at most $S(n)$ of those edges are needed to connect u to the selected set.

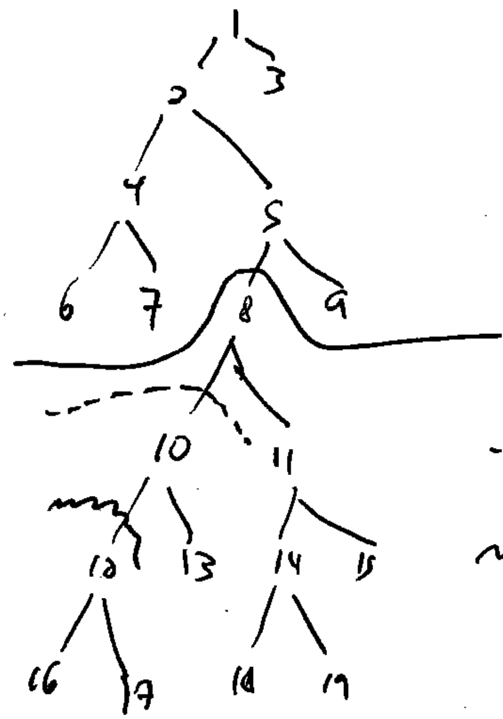
The total number is then no more than $S(n) + S(\frac{2}{3}n) + S(\frac{4}{9}n)$
$= \Gamma_S(n)$.

6.896

18.7

4-21-04

Example: Binary trees are 1-separable $\Rightarrow$ they are also strongly $\log_{3/2}$-separable.



~~Break into 10 + 9 nodes,~~

Select 10 nodes

select top half + need to select 2 nodes

----- is too big, select neither

〰 is just right

~~3 nodes are cut~~

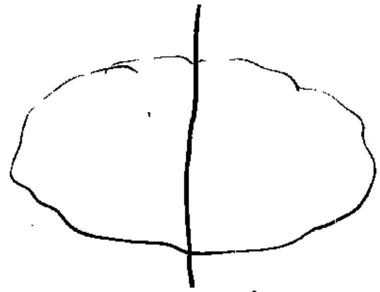
3 edges cut if first 10 nodes

$\Gamma_1(19) = \lceil \log_{3/2} 20 \rceil = 8$ so we did ok.

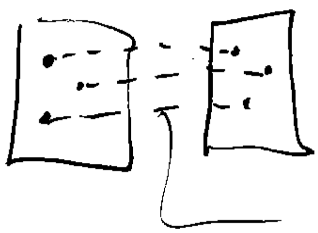
6.896
18.8
4-21-04

Back to layout algorithm

1) Separate
2) recurse
3) reassemble



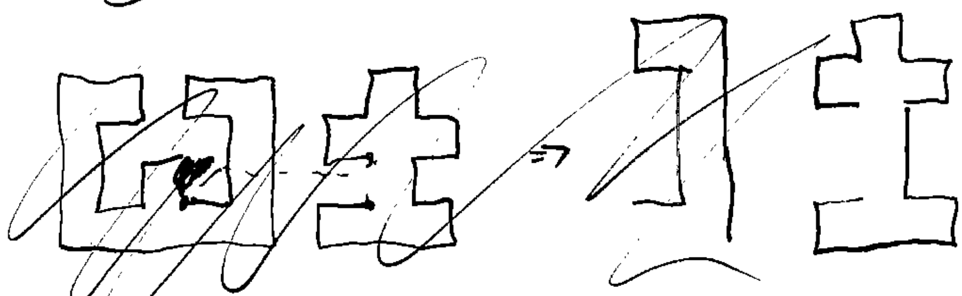
cut in 1/2 with separator
did recursive layout of halves
now what?



need to connect these edges.

e.g. ~~here is a tree~~

~~May have these recursive layout~~



insertChannels stretch layout vertically to make space
leave a space between for vertical routing

Example: $S(n) = O(1)$ then

$$\Gamma_S(n) = O(\log n)$$

Example: $S(n) = \lg n$ then

$$\Gamma_{\lg} n = \lg n + \lg \tfrac{2}{3} n + \lg \tfrac{4}{9} n + \cdots$$
$$= \lg n + \Gamma_{\lg}(\tfrac{2}{3} n)$$
$$= \lg^2 n$$

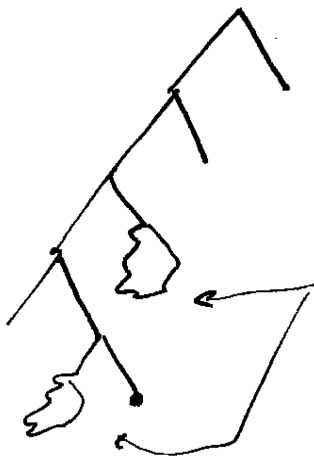
Lemma: if G is S-separable then G is strongly Γ_S separable.

~~proof: for any $t \leq |V|$ $\exists$ a path in the partition tree~~

Proof. Build a partition tree for G achieving S-separation

For any $t < |V|$ we will find ~~a collection of nodes in the partition tree~~ a path from the root of the partition tree to a leaf, & some subset of the siblings add up to exactly t nodes

e.g.



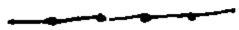
here is the P.T. and you can take any collection of the siblings. pick these two

to get exactly t

call these the selected sets

6.896
18.9
4-21-04

Example: layout of a linear array.

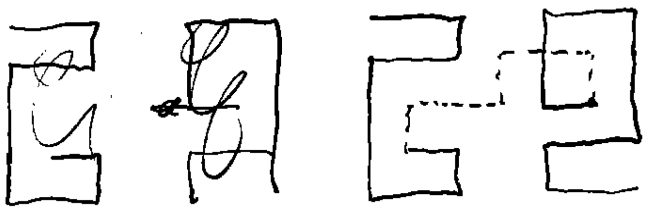


↑ two recursive layouts

---- need to connect

stretch vertically to bring ---- to edge

stretch horizontally between the halves



(n a power of 4)

Def'n $\Delta_S(n) = S(n) + 2S(n/4) + 4S(n/16) \cdots$

$= S(n) + 2\Delta_S(n/4)$

example: ~~S(n) = n^α then~~

[State as facts w/o proof]

$$\Delta_{n^\alpha}(n) = O(\sqrt{n}) \text{ if } \alpha < \tfrac{1}{2}$$
$$\Delta_{n^\alpha}(n) = O(n^\alpha) \text{ if } \alpha > \tfrac{1}{2}$$
$$\Delta_{n^{1/2}}(n) = O(\sqrt{n}\lg n)$$

$S(n) = n^\alpha$ $\Delta_S(n) =$

$$\Delta_S(n) = n^\alpha + 2\left(\frac{n}{4}\right)^\alpha + 4\left(\frac{n}{16}\right)^\alpha = O(\sqrt{n})$$

If $\alpha < \frac{1}{2}$ then the tops grow faster than the bottom

so $\Delta_S(n) = O(\sqrt{n})$

If $\alpha > 1/2$ then $\Delta_S(n) = O(n^\alpha)$

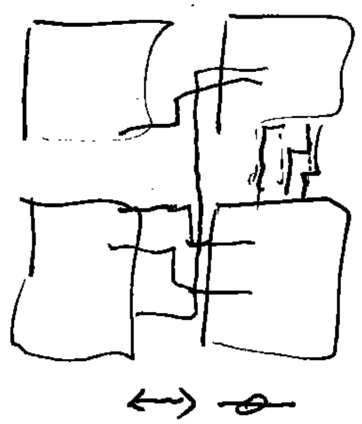
If $\alpha = \frac{1}{2}$ then $\Delta_S(n) = \Delta_{\sqrt{\ }}(n) = O(\sqrt{n}\lg n)$

6.896
18.10
4-21-04

Thm: $S(n)$ monotonically nondecreasing
A graph of n nodes with a strong ~~S(n) separator~~ S-separator can be laid out in a square with side length $O(\max(\sqrt{n}, \Delta_S(n)))$

Induction on n, assume n a power of 4
Claim side length of $\sqrt{n} + 6\Delta_S(n)$ good enough

base case: easy 1 node
induction: divide in half & half again



$$W(n) = 2\left(\frac{\sqrt{n}}{2} + 6\Delta_S(n/4)\right) + S(n)$$

but $\Delta_S(n) = S(n) + 2\Delta_S(n/4)$

$$= \sqrt{n} + 6\Delta_S(n/4) \quad \square$$

$$H(n) = S(n) + S(n/2) + 2H(n/4) = O(S(n)) + H(n/4)$$

Analysis: $S(n) = O(n^\alpha)$ for $\alpha < \frac{1}{2}$: $H(n) = O(\sqrt{n}) + H(n/4) = O(\sqrt{n} + \sqrt{\frac{n}{4}} + \sqrt{}$