DIVIDE-AND-CONQUER LAYOUT

A GENERAL LAYOUT ALGORITHM FOR LAYOUT PRODUCES LOW-DRAWN LAYOUT
FOR ANY ORDERED COMPLETE TREE OR PLANAR GRAPH,

A GENERAL LAYOUT ALGORITHM
NEAR OPTIMAL LAYOUT FOR ALL GRAPHS
OPTIMAL FOR BOUNDED DEGREE TREES + PLANAR GRAPHS

IDEA:

LAYOUT(G):
(1) FIND SMALL NUMBER OF EDGES THAT SEPARATE THE GRAPH INTO TWO NEARLY EQUAL PIECES. (CAN BE TOUGH)
(2) RECURSIVELY LAYOUT THE TWO HALVES
(3) ATTACH THE TWO HALVES TOGETHER, AND CARE IT UP.

DEFINITION:
G has an S-separator (i.e., S-separable) if
a) G has at least 1 vertex, or
b) G is a graph

- A set A \subseteq V(G) such that |A| < \frac{1}{2}|V(G)|
- (V(G) \setminus A) is two disconnected subgraphs

\begin{align*}
G_1 &= (V_1, E_1) \\
G_2 &= (V_2, E_2)
\end{align*}

\text{s.t. } |V_1| \geq \frac{1}{2}|V(G)|, \quad \text{and } |V_2| \geq \frac{1}{2}|V(G)|

\text{such that } G_1 \cup G_2 \text{ is the original graph}

\text{and } G_1 \text{ and } G_2 \text{ are } S\text{-separable.}
Example:

Then: Binary trees are 1-separate.

proof: pick a root
travel down the tree looking for a node that is
the ancestor of at most \( \frac{\sqrt{3}}{2} \) to \( \frac{2}{3} \) nodes.

that makes parent-child distance the key

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case A: this subtree is boxe \( \frac{2}{3} + \frac{3}{2} \) nodes

due

---

case B: subtree too small. Not of

Subtree too big. > \( \frac{3}{2} \) nodes

One or two children is at most half \( \frac{3}{4} \) the n. 85

So go to step

---

case C: too small. Not don't go there.

Early on...

1 command 19 nodes -- too big

2 commands 17 nodes too big

5 commands 13 nodes, too big

8 commands 12 nodes -- ok

Top left is 8 nodes

cut here -------- into 3 + 5

top 1/8 in 5 nodes

Top 1/8 in 5 nodes

cut here ---

And so forth

SAVE THIS TREE
Defn:
A path-finite tree of a GBA is a tree where each node is labeled by a set.

Example:
\[
\{1,2,3,5,9\} \cup \{4,6,7\} \supset \{1,3,9\} \cup \{2,5,9\}
\]
\[
\{8,10,19\} \cup \{8,19\}
\]

It's a tree even if G is not a tree.
Example: A grid of squares is \( O(V^2) \) separable.


![Grid Diagram]

**Theorem:** A grid is \( O(V^2) \) separable.

**Definition:** A grid is \( S \)-separable if the sizes of its subgraphs are at most \( \frac{W+1}{2} \).

[Cut exactly in half area, otherwise a close as possible]

**Example:** A grid of size \( 2 \times 2 \) is already \( S \)-separable.

**Claim:** not provable. We'll show \( S \)-separable \( \implies \) \( S \)-separable \( \implies \) stands separable.

**Definition:** \( \Pi_s(n) \) defined as

\[
\Pi_s(n) = S(n) + S\left(\frac{2}{3}n\right) + S\left(\frac{4}{9}n\right) + \ldots
\]

\[
= \sum_{i=0}^{\log_2 n} S\left(\frac{2^i}{n}\right)
\]

**Example:**

\[
S(n) = n^d \implies \\
\Pi_s(n) = n^d + \left(\frac{2}{3}n\right)^d + \left(\frac{4}{9}n\right)^d + \ldots
\]

\[
\leq \frac{n^d}{1 - (\frac{2}{3})^d} = O(n^d)
\]
proof by induction: \( \forall n \leq t < |V| \)

if \( n, t \leq t \) then add to left subtree to our selected set + go right to \( n, t - n, \) etc.

if \( n, t + t > t \) then prunese
if \( n, t \) then go left
don't we the right subtree in
pick \( n, \) art's from left.

claim: the edge connects our selected set to anything else
\( \leq g(n) \) edges connecting selected set to anything else.

pf.

if \( u \) is selected then at most \( g(n) \) edges connect \( u \) to non-selected set.
if \( u \) is not selected then at most \( g(n) \) edges connect \( u \) to any \( v \) to the selected set.

if \( u \) is selected then at most \( g(n) \) or to edges
do \( u \) are needed to connect \( u \) to non-selected node(s).

if \( u \) is not selected, then at most \( g(n) \) of these edges are needed to connect \( u \) to the selected set.

The total number is then \( g(n) + g(\frac{g(n)}{2}) + g(\frac{g(n)}{4}) = g(n) \).
Example: Binary trees are 1-separated $\Rightarrow$ they are strongly 1-separated.

$0.896$

$18.7$

$4.31.09$

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Deuce into 10 + 4 nodes,

Select 10 nodes

Select top half \textbf{and} need to select 2 nodes

----- is too big, select next for

\textbf{new is just right}

Bodies cut if less than 10 nodes

\[ \Gamma_1(19) = \Gamma_{15^2}, \quad 20 \cdot 7 = 8 \quad \text{so we did so.} \]
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Back to layout algorithm
1) separate
2) reverse
3) reassemble

Cut in 1/2 with opposite diagonal line first or 6, 1/2 now what?

Need to correct free edge.

0.7

Here is a tree
My face is on the back

Insert heading, stretch right vertically to make sure there's space between for vertical rule.
Example 6: 
\[ S(n) = O(1) \quad \text{for} \]
\[ \Gamma_{S}(n) = O(\log n) \]

Example 7: 
\[ S(n) = \log n \quad \text{to} \]
\[ \Gamma_{\log n} = \log n + \log^{2} n + \log^{3} n + \ldots \]
\[ = \log n + \Gamma_{\log (\log n)} \]
\[ = \log n \]

Lemma: If \( G \) is \( S \)-separable then \( G \) is strongly \( \Gamma_{S} \)-separable.

Proof: For any \( t \leq |V| \) there is a path in the partition-tree

For any \( t \leq |V| \) we will find a collection of nodes in the partition tree. A path from the root of the partition tree to a leaf, and some subset of the siblings (up to exactly \( t \) nodes)

E.g.

Here is an example. Any path can be any collection of siblings. Pick one for two to set \( t \) to separate sets.

Call these \( t \) selected sets.
Example: let $X$ be a linear array.

\[
\begin{align*}
\Delta_5(n) &= S(n) + 2S(n/4) + S(n/16) \\
&= S(n) + 2\Delta_5(n/4)
\end{align*}
\]

(n a power of 4)

Definition

\[\Delta_5(n) = S(n) + 2S(n/4) + S(n/16) \ldots\]

Example:

\[S_{\Delta_5}(n)\]

\[\Delta_5(n) = n^d + 2\left(\frac{n}{4}\right)^d + 4\left(\frac{n}{16}\right)^d = \Theta(n^{d+1})\]

If $\alpha < \frac{1}{2}$, then $\Delta_{\Delta_5}(n)$ grows faster than $\Delta_5(n)$.

\[\Delta_{\Delta_5}(n) = \Theta(n^{d+2})\]

If $\alpha > \frac{1}{2}$, then $\Delta_5(n) = O(n^d)$.

If $\alpha = \frac{1}{2}$, then $\Delta_5(n) = \Delta_{\Delta_5}(n) = O(n \log n)$.
Then: $S(n)$ monotonically non-decreasing.

A graph with a strong $S(n)$-separator $S$-separator can be laid out in a space with side length $O\left(\max(\sqrt{n}, \Delta_S(n))\right)$

Induction $n$, assume $n$ a power of 4

Claim: side length is $\left(\frac{n}{4} + 6\Delta_S(n)\right)$ good enough

Base case: easy to see

Induction: divide in half + half can

\[ W(n) = \sqrt{\frac{n}{3}} + 6\Delta_S(\frac{n}{4}) + S(n) \]

But $\Delta_S(n) = S(n) + 2\Delta_S(\frac{n}{4})$

$$= \sqrt{n} + 6\Delta_S(\frac{n}{4})$$

$$W(n) = S(n) + S(\frac{n}{4}) + 2H(\frac{n}{4}) = O(S(n)) + H(\frac{n}{4})$$

Analysis: $S(n) = O(n^\alpha)$ for $\alpha < \frac{1}{2}$:

$$H(n) = O(\sqrt{n}) + H(\frac{n}{\sqrt{2}}) = O(\sqrt{n} + \sqrt{\frac{n}{\sqrt{2}}} + \sqrt{\frac{n}{\sqrt{8}}})$$