Hypercube network

d dimensions
N = 2^d nodes

\begin{itemize}
  \item d=0
  \item N=1
  \item d=1
  \item N=2
  \item d=2
  \item N=4
  \item d=3
  \item N=8
  \item d=4
  \item N=16
\end{itemize}

Label each of the 2^d nodes with a d-bit binary string:
\[ b_{d-1} b_{d-2} \ldots b_0 \]

Connect two nodes if they differ in exactly 1 bit:
\( b_{d-1} b_{d-2} \ldots b_0 \)
\[ \text{connected to} \]
\[ b_{d-1} b_{d-2} \ldots b_0 \]
\[ b_{d-1} b_{d-2} \ldots b_0 \]
\[ \vdots \]
\[ b_{d-1} b_{d-2} \ldots b_0 \]

Diameter = \( d = \lg N \)
Degree = \( d = \lg N \)
\( BW = N/2 \)
\# wires = \( Nd/2 = \Theta(N \lg N) \)
Embeddings in the hypercube

Theorem The \( N \)-node hypercube contains an \( N \)-node linear array as a subgraph (i.e., a Hamiltonian path).

Pf. True for \( N = 1, 2, 4 \):

\[
\begin{array}{c}
O \quad O \quad O \\
\end{array}
\]

Induction on \( d \). Claim \( \exists \) Hamiltonian cycle for \( d \)-dim hypercube for \( d \geq 2 \).

Base:

\[
\begin{array}{c}
\end{array}
\]

Assume claim true for \( N/2 \)-node hypercube.
Consider \( N = 2^d \) hypercube.

Consists of 2 \( N/2 \)-node hypercubes containing identical Hamiltonian cycles (by IH). Let \((Ox_1, Ox_2)\) be any edge in 1st subcube that cycle goes through, and let \((1x_1, 1x_2)\) be corresponding edge in 2nd subcube. Replace these two edges with \((Ox_1, 1x_1)\) and \((Ox_2, 1x_2)\). \( \Box \).
A d-bit Gray code is an ordering of the $2^d$ d-bit bit-strings such that each string differs from the previous in exactly one bit.

Ex. $d=3$

<table>
<thead>
<tr>
<th>0</th>
<th>000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>001</td>
</tr>
<tr>
<td>2</td>
<td>011</td>
</tr>
<tr>
<td>3</td>
<td>010</td>
</tr>
<tr>
<td>4</td>
<td>110</td>
</tr>
<tr>
<td>5</td>
<td>111</td>
</tr>
<tr>
<td>6</td>
<td>101</td>
</tr>
<tr>
<td>7</td>
<td>100</td>
</tr>
</tbody>
</table>

"Reflecting" Gray code

Corollary. d-bit Gray codes exist for all $d$.

Hamiltonian path in hypercube = Gray code.

Theorem. Let $d_1 + d_2 \leq d$. Then a $2^{d_1} \times 2^{d_2}$ mesh (or torus) can be embedded in an $N = 2^d$-node hypercube.

Pf. Let $g_1(x_1)$ be $d_1$-bit Gray code of $x_1$, where $0 \leq x_1 < 2^{d_1}$.

Let $g_2(x_2)$ be $d_2$-bit Gray code of $x_2$, where $0 \leq x_2 < 2^{d_2}$.

Map node $(x_1, x_2)$ of mesh to node $g_1(x_1) \parallel g_2(x_2)$ of hypercube.

Ex. $8 \times 8$ mesh.

$$(3, 6), (4, 5), (4, 6), (4, 7), (5, 6)$$

Corollary. $2^{d_1} \times 2^{d_2} \times \cdots \times 2^{d_k}$ mesh embedded in $2^{d_1 + d_2 + \cdots + d_k}$ hypercube.

Fact: $3 \times 5$ mesh cannot be embedded in 16-node hypercube. But, $m \times n$ mesh can be embedded in $2^{\log(mn)}$-node hypercube with dilation 2.
Embedding trees in hypercubes

Thm. Not possible to embed \((N-1)\)-node complete binary tree in \(N\)-node hypercube.

Proof. Suppose possible. Root mapped to node 00...0.
Depth-1 nodes mapped to nodes with odd parity.
Depth-2 nodes mapped to nodes with even parity.
Depth-3 nodes mapped to nodes with odd parity.

(Def. Parity = odd if \# 1's is odd.
\(\text{even if \# 1's is even.}\)

\# leaves = \(N/2\) : all have same parity.
\# grandparents of leaves = \(N/8\) : same parity as leaves.

But, hypercube has \(N/2\) nodes with even parity
and \(N/2\) nodes with odd parity, and tree must have \(\geq N/2 + N/8\) nodes with same parity.

Def. Double-rooted complete binary tree:

\[\text{Diagram of double-rooted complete binary tree.}\]

Thm. \(N\)-node double-rooted cbt is subgraph of \(N\)-node hypercube, for \(N \geq 2\).

Proof. Induction on \(d\).
\[d = 1 \quad (N = 2):\]

\[d = 2 \quad (N = 4):\]

\[\text{Diagram of double-rooted complete binary tree.}\]
$d \geq 3$ ($N \geq 8$): Embed double-rooted cft on $N/2$ nodes in $N/2$-node O-subcube. Consider top 4 nodes:

\[
\begin{array}{cccc}
00000 & 00100 & 00101 & 01001 \\
\end{array}
\]

WLOG, $a = 00\ldots0$
a, b differ in dim $i_0$
b, c differ in dim $i_1$
c, d differ in dim $i_2$

Note: $i_0 \neq i_1 \neq i_2$ (or else $a = c$ or $b = d$).

Embed double-rooted cft on $N/2$ nodes in $N/2$-node 1-subcube identically, except $b' = 100\ldots0$ and permute dimensions $i_i \rightarrow i_0$ and $i_2 \rightarrow i_1$.

Thus, $(a, b'), (b, c'),$ and $(c, d')$ adjacent.

b, c' new roots
Corollary: \((N-1)\)-node CBT embeds in \(N\)-node hypercube with dilation 2.

**Proof:**

Embed CBT into double-rooted CBT with 1 edge having dilation 2.

Fact: All \(N\)-node binary trees can be embedded into \(N\)-node hypercube with \(O(1)\) dilation.