The Spin-Block Problem

A process waiting for a lock can

- spin — cost = time spinning.
- block — fixed cost = C.

Continuous version of ski rental problem.

Recall: Deterministic 2-competitive alg:

Spin for C steps. Then block. (Best possible deterministically)

Today: Randomized algo that beat 2.

Simpler Alg:

Spin until time $C/2$.

With prob p, block.

With prob $1-p$, keep spinning until time C. Then block.

Question: What is optimal C?

Defn: $f(t)$ = expected cost.

$$f(t) = \begin{cases} t & (t < c/2); \\ t + pc & (c/2 \le t < c); \\ p\left(\frac{3c}{2}\right) + (1-p)2c & (t \ge c). \end{cases}$$

Goal: Choose p to minimize competitive ratio:

(1) $\quad f(t) \le (1+\alpha)t \qquad (c/2 \le t < c)$

(2) $\quad f(t) \le (1+\alpha)c \qquad (t \ge c)$.

Worst case for (1): $\quad \frac{t+pc}{t} \le (1+\alpha)$ when $t = c/2$.

Set inequalities to equalities:

$$f(t) = c/2 + pc = (1+\alpha)\, c/2$$

$$f(t) = p(3c/2) + (1-p)2c = (1+\alpha)c$$

$$\frac{c/2 + pc}{c/2} = \frac{\frac{3pc}{2} + 2(1-p)c}{c}$$

$$\cancel{1+} \; 2p = \frac{3p}{2} + \cancel{2} - 2p \quad -\tfrac{1}{2}p + 1$$

$$p = 2/5.$$

$$\Rightarrow \quad f(t) = \frac{c/2 + 2/5\,c}{c/2} = 9/5 = 1.8 = (1+\alpha).$$

$\Rightarrow$ Competitive ratio $(1+\alpha) = 1.8$.

Less than 2!!

SANITY check:

Def: Function $\pi(t)$ — density function of time before a process should block.

Expected cost of waiting q steps

$$f(q) = \int_{t=0}^{q} dt\ (t+c)\ \pi(t) + q \int_{t=q}^{\infty} dt\ \pi(t).$$

Idea: Choose $\pi(t)$ to minimize comp ratio:

$$f(q) \le (1+\alpha)\ q \qquad (q < c);$$
$$f(q) \le (1+\alpha)\ C \qquad (q \ge c).$$

Set inequalities to equalities.

$$f(q) = (1+\alpha)\ q \qquad (q < c);$$
$$f(q) = (1+\alpha)\ C \qquad (q \ge c).$$

Differentiate $f(q)$:

$$f'(q) = (q+c)\ \pi(q) + \int_{t=q}^{\infty} dt\ \pi(t) - q\ \pi(q).$$

$$= c\,\pi(q) + \int^{\infty} dt\ \pi(t).$$

Differentiate again:

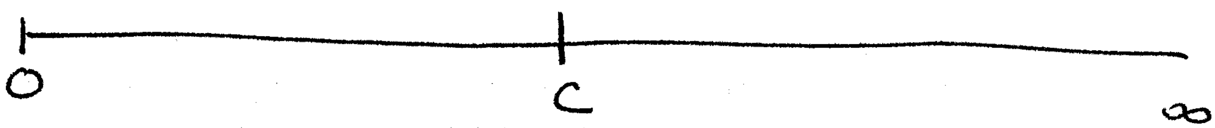
$$f''(q) = c\pi'(q) - \pi(q)$$

Note: $f''(q) = 0$. $(q \geq 0)$.

Thus,

$$\pi(q) = c\pi'(q)$$
$$= A e^{q/c}, \text{ for some const } A.$$

Note: A different in each range:



Recall: $$\int_{t=c}^{\infty} dt\, \pi(t) = \int_{t=c}^{\infty} dt\, A\, e^{t/c}$$

$$\leq 1$$

$$\Rightarrow A = 0 \qquad (t \geq c).$$

Thus, $$\int_{t=0}^{c} dt\, \pi(t) = \int_{t=0}^{c} dt\, A e^{t/c}$$

$$= 1$$

$$\int_{t=0}^{c} A e^{t/c}\,dt = Ac\, e^{t/c}\Big|_0^c$$

$$= Ac\,(e-1)$$

$$A = \frac{1}{c(e-1)}.$$

$$\pi(t) = \begin{cases} \frac{1}{c(e-1)}\, e^{t/c} & (0 \le t < c) \\ 0 & (t \ge c) \end{cases}$$

Just calculated $\pi(t)$. Now calculate comp rates (i

$$(1+\alpha) - \frac{f(c)}{c} = \frac{1}{c} \int_{t=0}^{c} dt \, (t+c) \, \underset{\displaystyle A e^{t/c}}{\underset{\shortparallel}{\pi(t)}}$$

$$= \frac{1}{c} \cdot \frac{1}{c(e-1)} \int_{t=0}^{c} dt \, (t+c) \, e^{t/c}$$

$$= \left(\frac{1}{c^2(e-1)} \right) \left[(t+c) \, c e^{t/c} \Big|_0^c - \int_0^c c e^{t/c} \, dt \right]$$

$$= \left(\frac{1}{c^2(e-1)} \right) t c e^{t/c} \Big|_0^c$$

$$= \frac{e}{e-1}$$

Discrete case:

$$\pi_i = \begin{cases} \frac{\alpha}{P}\left(\frac{P+1}{P}\right)^{i-1} & i = 1 \cdots P \\ 0 & \text{o.w.} \end{cases}$$

~~[illegible]~~

$$1+\alpha \longrightarrow \frac{e}{e-1} \approx 1.58$$