6.895 Theory of Parallel Systems	Lecture 18

Hypercube Networks-III

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Lecture Summary

1. *Review of the previous lecture*
   This section highlights the worst case routing on the butterfly network.
2. *Valiant's Algorithm*
   This section describes the idea behind Valiant's algorithm for solving a hard problem on the butterfly network.
3. *Ranade's Algorithm*
   This section briefly introduces Ranade's algorithm.
4. *Shuffle-exchange graph*
   This section explains the properties of the Shuffle-exchange graph.
5. *DeBruijn Network*
   This section introduces the DeBruijn Network, and illustrates the DeBruijn sequence with the aid of a card trick.
6. *Bisection Width for Shuffle-Exchange graph*
   This section provides the proof for deriving the bisection width for the shuffle-exchange graph.

1 Review of the previous lecture

- Routing on an N-node (i.e. $N/\lg N$ input)butterfly takes $O(\lg N)$ time with high probability, assuming all routing problems are equally likely.
- Some problems are still hard. For example, the Matrix transpose problem is hard. If we consider the routing from $X_1X_2X_3X_40000 \rightarrow 0000X_1X_2X_3X_4$, essentially $\sqrt{n}$ inputs have to route through a common intermediate node, 00000000. This routing step is a subproblem of the matrix transpose problem wherein $(i,j) \rightarrow (j,i)$. The indices i and j are represented by bits and are concatenated to form the input. Therefore, matrix transpose incurs $\sqrt{n}$ congestion. Hence, we find that the butterfly network has a bad worst case O($\sqrt{n}$), even though it has a good average case.

2 Valiant's Algorithm

Valiant's Algorithm is a *permutation routing* algorithm. The idea is to convert one hard problem into two easy ones. The routing is performed in two phases:

- **Phase 1:** Firstly, we route from the sources to random intermediate nodes.
- **Phase 2:** Then, we route from the intermediate nodes to the destination nodes.

Both phases 1 and 2 are *random routing problems* which are good cases for the butterfly network. Hence, there is no hard routing problem. We can however be *unlucky*, but it is quite *unlikely* that we are unlucky. The *unluckiness* does not correspond to the choice of input because we cannot put a bound on the choice of input. However, we can bound probabilistically how likely we could have a bad routing problem. An adversary cannot elicit the worst case behavior. In fact, the adversary cannot provide a bad routing problem because every problem is equally bad. In conclusion, by randomizing the chosen intermediate nodes which are destinations in Phase 1 and sources in Phase 2, we have essentially distributed the congestion, thereby making the sub-problems easy ones.

3 Ranade's Algorithm

It is a routing algorithm which runs in $O(\lg N)$ time with high probability. It has some interesting properties like bounded queues, which are queues of size O(1). The proof is through means of an elegant delay sequence argument.

4 Shuffle-exchange graph

Description

Definition 1 (Shuffle Exchange Graph) *Let $d \epsilon N$. The d-dimensional shuffle-exchange SE(d) is defined as an undirected graph with node set $V = [2]^d$ and an edge set $E = E_1 \cup E_2$ with*

$E_1 = \{((a_{d-1}, ..., a_0), (a_{d-1}, ..., \overline{a_0}))|(a_{d-1}, ..., a_0)\epsilon[2]^d, \overline{a_0} = 1 - a_0\}$

and

$E_2 = \{((a_{d-1}, ..., a_0), (a_0, a_{d-1}, ..., a_1))|(a_{d-1}, ..., a_0)\epsilon[2]^d\}$.

It is based on an N node hypercube, where $N = 2^d$. Figure 1 shows an 8-node shuffle-exchange graph. Each

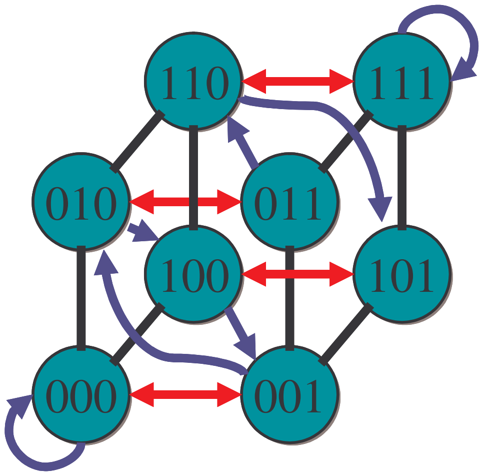


Figure 1: Shuffle-exchange graph

node is labelled with a unique $\lg N$ bit string. A node labelled a $= a_{\lg N-1}, ..., a_0$ is linked to a node labelled b $= b_{\lg N-1}, ..., b_0$ by a *shuffle edge* if rotating one position to the left or right yields b, i.e., if either b $= a_0, a_{\lg N-1}, a_{\lg N-2}, ..., a_1$ or b $= a_{\lg N-2}, a_{\lg N-3}, ..., a_0, a_{\lg N-1}$.Two nodes labelled a and b are linked by an *exchange edge* if a and b differ in only the least significant (rightmost) bit, i.e., b $= a_{\lg N-1}, a_{\lg N-2}, ..., \overline{a_0}$. In Figure 1, shuffle edges are the blue edges, and exchange edges are the horizontal red edges. Figure 2

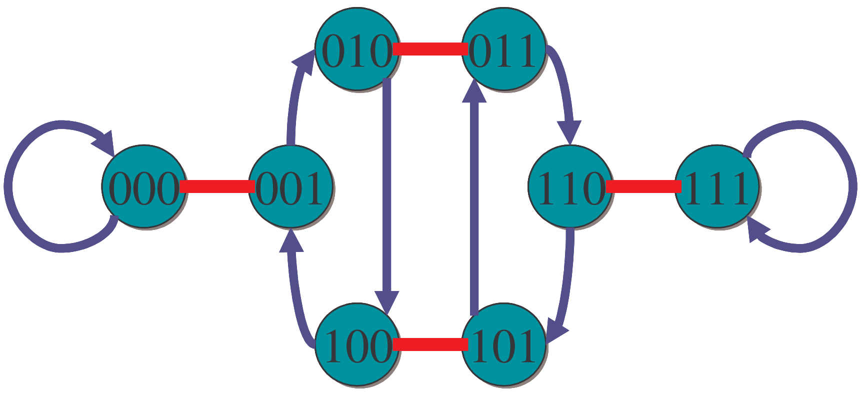


Figure 2: Shuffle-exchange graph: Another view

shows another view of the same 8 node shuffle-exchange graph. To summarize, if a node is a d-bit binary number, *exchange edges* are between $< b_{d-1}, b_{d-2}, ..., b_1, 0 > \leftrightarrow < b_{d-1}, b_{d-2}, ..., b_1, 1 >$ and *shuffle edges* are from $< b_{d-1}, b_{d-2}, ..., b_1, b_0 > \rightarrow < b_{d-2}, ..., b_0, b_{d-1} >$, which corresponds to a left-cyclic shifting of the bits.

Properties

- Degree = 3 (Out-degree = 2 , In-degree = 2)
- Number of edges $\leq$ 2N(in the case of an undirected graph). According to the Hand-shaking lemma, sum of the in-degrees = number of edges. Since there are N nodes, each of which has an in-degree of at most 2, the total number of edges would be at most 2N.
- Diameter $\leq$ 2 lg N. Since there are $\lg N$ bits in a node, in order to route from one node to another, even in the worst case would correspond to traversing at most $\lg N$ shuffle edges and $\lg N$ exchange edges. Hence the diameter $\leq$ 2 lg N.
- Bisection Width = $O(N/\lg N)$. Deriving the Bisection Width is not trivial as is illustrated by the proof in Section 6.

It can be seen that every property of the butterfly is also a property of a shuffle-exchange graph. However, the properties were easier to prove in the case of the butterfly because it was more intuitive.

A Perfect (Out) Shuffle

The connection pattern in Figure 3 is called a *perfect shuffle* because it is like shuffling two halves of a card deck, so that the cards from the two decks interleave perfectly. We notice that the top and bottom cards which correspond to bit patterns 000 and 111 do not change positions, whereas the other positions go a position which is double in value, that also corresponds to a left shift of one bit [1]. For example, 001 goes to 010 that is double of it. Hence, we can conclude that **Shuffle(X) $\equiv$ 2X mod (N-1)**. This shuffle, wherein the top and bottom cards retain their positions is called a *Perfect Out-Shuffle*.

A Perfect (In) Shuffle

In a perfect in-shuffle as shown in Figure 4 the top and the bottom cards are *in* the deck rather than *out*, which means that they do not retain their positions.

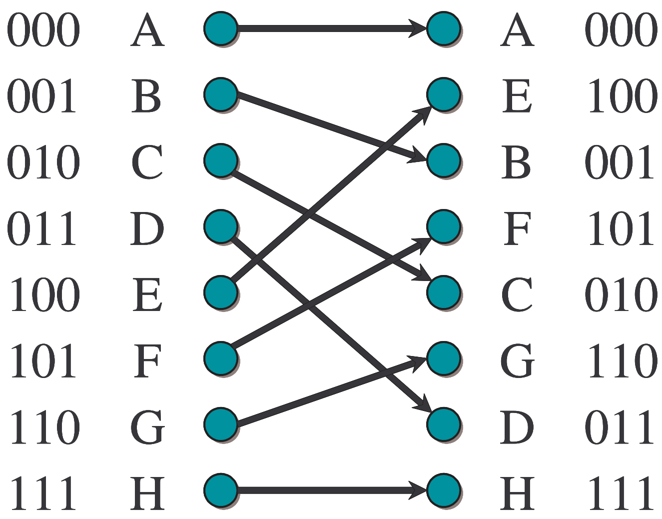


Figure 3: A perfect shuffle

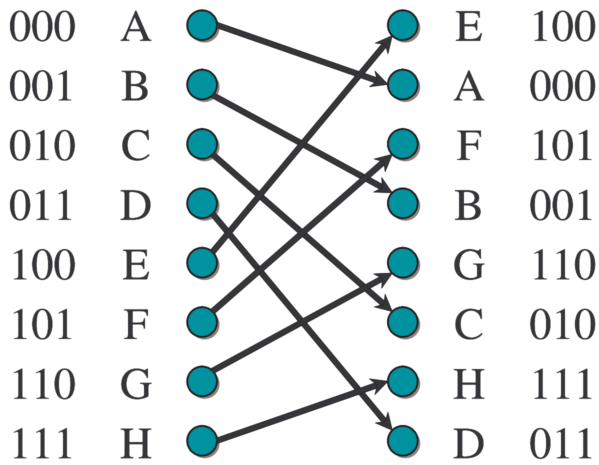


Figure 4: A perfect in-shuffle

A perfect in-shuffle can be viewed as a perfect out-shuffle followed by an exchange, which is illustrated in Figure 5.

A Card trick

Consider a deck of 8 cards which correspond to the 8 nodes shown earlier in the shuffle-exchange graph. Suppose a card is chosen at random and placed in any location in the deck, given a required destination location, we can perform a certain number of perfect shuffles(in or out) in order that we route the card to the required destination. Let us consider an example. Suppose that the randomly chosen card is placed at the last position (corresponding to 8-1 = 7 = 111), and that the required destination position is 3 (corresponding

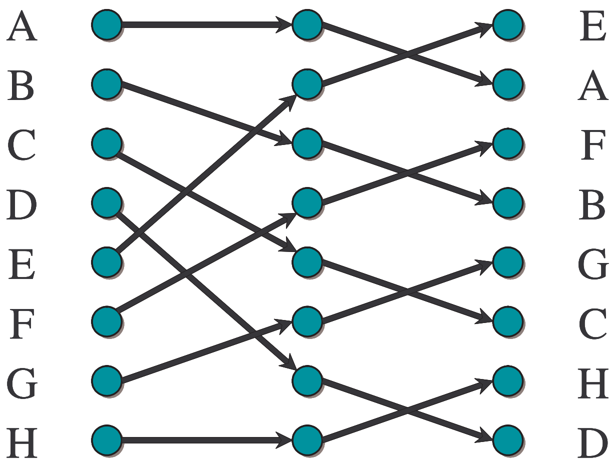


Figure 5: In-shuffle = Out-shuffle + Exchange

to 3-1 = 2= 010). The claim is that 3 perfect shuffles is sufficient to place the last card in the third position. It can be seen from Figure 6 that starting with an in-shuffle, followed by an out-shuffle, and lastly performing an in-shuffle again would route the card at 111 to the position 010. Let us proceed to trace through the sequence of operations. The first in-shuffle actually corresponds to a shuffle followed by an exchange. We take the redundant shuffle edge from 111 we go back to 111 and then take the exchange edge to 110. The second operation which is an out-shuffle just corresponds a shuffle from 110 which takes us to 101. The last operation which is an in-shuffle takes us from 101 to 011 following the shuffle edge and then from 011 to the required position 010 following the exchange edge. This sequence is illustrated by the green edges in Figure 6. In order to obtain the required sequence of operations, we first take the XOR of the source and the destination. The bits which are 1 in the resultant bit string correspond to the bits that have to be inverted by taking exchange edges at appropriate rotations. Hence, we perform an in-shuffle for each 1 in the bit string, and perform an out-shuffle for each 0 in the resultant bit-string.

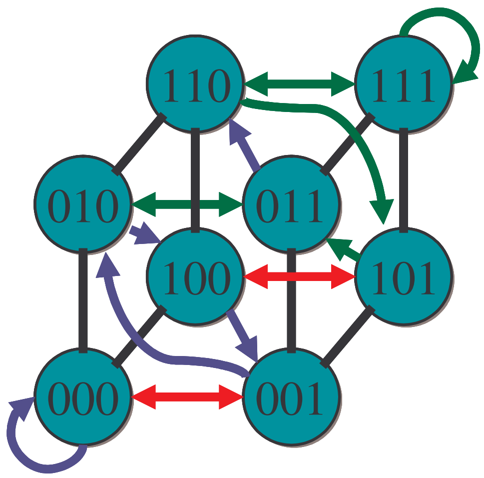


Figure 6: Required path in the shuffle-exchange graph (Shown in green)

5 DeBruijn Network

In this section, we look at the ***DeBruijn graph***. This network is based on the Hypercube. In fact, we may convert the previous Shuffle Exchange graph to the DeBruijn graph by contracting the exchange edges. Consider a 3-bit node as an instance. We use the first 2 bits as the prefix, and then combine all nodes with the same prefix. For instance, node $\langle 1,1,0\rangle$ and $\langle 1,1,1\rangle$ will merge into one node $\langle 1,1\rangle$, and so on. Then we collapse the Shuffle-exchange network along the exchange edges to obtain the DeBruijn Network (as shown in Figure 7).

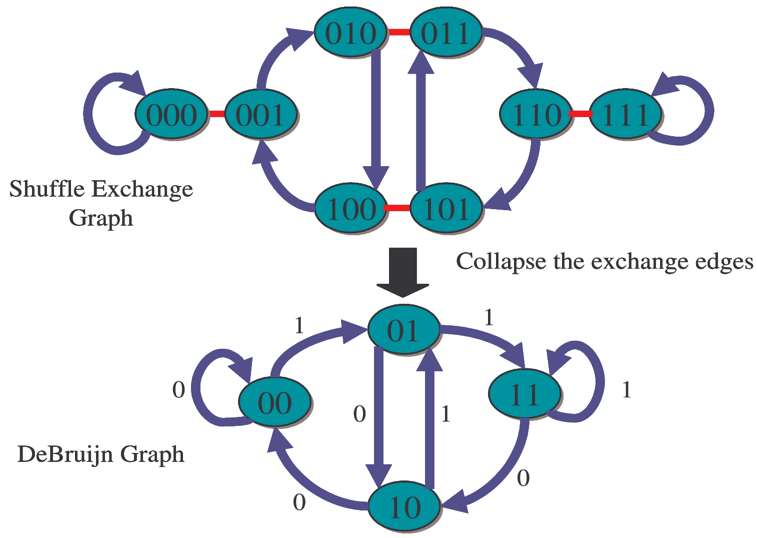


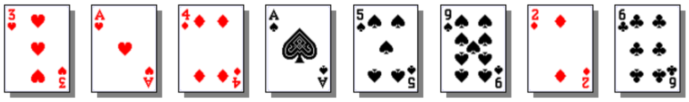
Figure 7: Converting the Shuffle-exchange graph to the DeBruijn graph

In one DeBruijn network, we have:
Node = d-bit binary number
Out-shuffle edge $\langle b_{d-1}, b_{d-2}, ...b_0\rangle \rightarrow \langle b_{d-2}, ...b_0, b_{d-1}\rangle$
In-shuffle edge $\langle b_{d-1}, b_{d-2}, ...b_0\rangle \rightarrow \left\langle b_{d-2}, ...b_0, \overline{b_{d-1}}\right\rangle$
The shuffle edges can also be represented as 0-edges or 1 edges based on whether the higher order bit, b_{d-1} is a 1 or a 0, like:
0-edge is $\langle b_{d-2}, ...b_0, 0\rangle$
1-edge is $\langle b_{d-2}, ...b_0, 1\rangle$

Degree = 4 (out-degree = in-degree = 2)
Number of edges $\leq$ 4N
Diameter = lgN
Bisection width = O(N / lgN).
We had shown that an (n-1)-dimension DeBrujin network is obtained by collapsing the exchange edges of an n-dimensional S-E network, which has a bisection width of O(N/lgN) (which will be proved in Section 6). Since the DeBrujin network is a reduced form of the S-E network, the DeBrujin network has the same upper bound of the bisection width = O(N/lgN).

Another Card Trick
This card trick is related to DeBruijn networks. We consider a deck of 8 cards. Suppose that the deck is

cut 3 times, and then 3 cards are chosen consecutively. Now, if the number of black cards chosen is known, then we can find which 3 cards were chosen ! Knowing the number of black cards provides us 3 bits of information, and it is only 3 bits of information that we need in order to get the position of a given card. According to the listing of the 3 card sequences in Figure 8, we find that any sequence of 3 cards corresponds to a different pattern of cards. Any kind of pattern only occurs once there. So 3 bits information is enough for us to predicate what cards are in hand. For example, if the pattern is 100, then it shows that the cards in hand are 6 clubs, 3 hearts and A hearts. By performing an Eulerian tour of the DeBruijn network in Figure 7, one of the DeBruijn sequences we get is shown in Figure 8.



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0001110100
000
 001
  011
   111
    110
     101
      010
       100
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Figure 8: Another Card trick (DeBruijn Sequence)

6 Deriving the Bisection Width for the Shuffle-Exchange Graph

In this section, we will prove that the bisection width of S-E network is O(N/lgN). The proof is based on the complex-plane diagram of the Shuffle Exchange network (as shown in Figure 9). The dotted edges are the exchange edges and the solid edges stand for the shuffle edges.
$N = 2^d$
Let w_d=$e^{2\pi i/d}$ be d-th complex root of unity. Hence, w_d^d=1

Definition 2 $\sigma(b_{d-1}, b_{d-2}, ...b_0) = \sum_{k=0}^{d-1} b_k w_d^k$

A few lemmas which describe the properties of the S-E network are described below. The lemmas form the crux of the proof.

Lemma 3 *Exchange edges have unit length.*

Proof The least significant bits of nodes connected by exchange edges are complements of one another.

$$\sigma(b_{d-1}, b_{d-2}, ...1) = \sigma(b_{d-1}, b_{d-2}, ...0) + 1w_d^0 \tag{1}$$

□

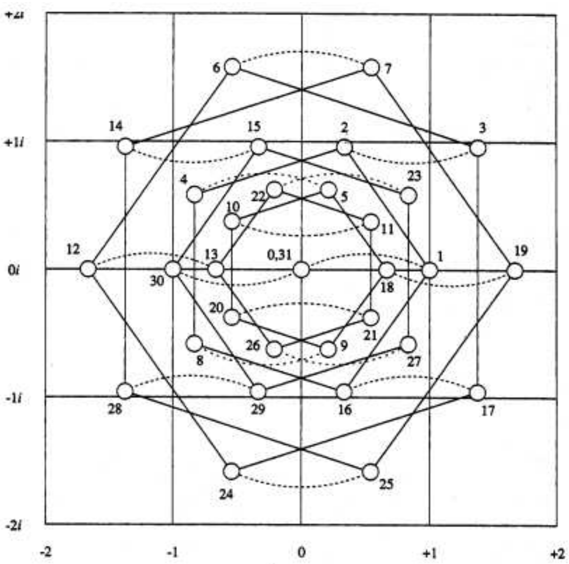

Figure 9: Complex plane diagram of the Shuffle-exchange network

Lemma 4 *Shuffle edges form* necklaces*, all of whose nodes are equidistant from origin:*

Proof

$$w_d\sigma(b_{d-1}, b_{d-2}, ...b_0) = w_d \sum_{k=0}^{d-1} b_k w_d^k = \sum_{k=0}^{d-1} b_k w_d^{k+1} = \sigma(b_{d-2}, ...b_0, b_{d-1}), since\ w_d^d = 1 \quad (2)$$

Thus, $\|\sigma(b_{d-1}, b_{d-2}, ...b_0)\| = \|\sigma(b_{d-2}, ...b_0, b_{d-1})\|$.

Most nodes are in *full* necklaces , which have d nodes, but some are in *degenerate* necklaces, which have lesser nodes.

For example:
For N=16 : {0000}, {1111}, {0101,1010} are *degenerate* necklaces.

□

Lemma 5 *All degenerate necklaces are mapped to origin.*

Proof Suppose: $w_d^k\sigma(x) = \sigma(x)$ with k<d
Each multiplication by w corresponds to a shuffling which in turn rotates the necklace (1/d)th each time.
Since we have $w_d^k \neq 1$ (k<d), we infer that $\sigma(x)$=0.

□

Lemma 6 *O(N/lgN) nodes are mapped to origin.*

Proof

$$\sigma(b_{d-1}, b_{d-2}...b_0) = 0 \Rightarrow \sigma(b_{d-1}, b_{d-2}...\overline{b_0}) = \pm 1 \tag{3}$$

The necklace of $\langle b_{d-1}, b_{d-2}, ..., \overline{b_0} \rangle$ has lgN nodes and there are at most two of them lying at $\pm$ 1. Therefore, there are at most N/lgN full necklaces. This means that there are at most 2N/lgN nodes are at ± 1. Hence $\leq$ 2N/lgN nodes are at 0.

□

Lemma 7 *O(N/lgN) nodes are mapped to real line*

Proof At most 2 nodes from each full necklace and O(N/lgN) nodes of degenerate necklaces at origin map to the real line.

□

Lemma 8 *O(N/lgN) edges touch real line.*

Proof There are at most 4 shuffle edges per full necklace that corresponds to O(N/lgN) edges. In addition there are O(N/lgN) shuffle edges at origin and O(N/lgN) horizontal exchange edges. Hence, O(N/lgN) edges touch real line.

□

Lemma 9 *Real line forms bisection.*

Proof

$$\sigma(b_{d-1}, b_{d-2}...b_0) + \sigma(\overline{b_{d-1}}, \overline{b_{d-2}}...\overline{b_0}) = \sum_{k=0}^{d-1}(b_k + \overline{b_k})w_d^k = \sum_{k=0}^{d-1} w_d^k = \frac{w_d^d - 1}{w_d - 1} = 0,\ \textit{since}\ w_d^d = 1 \tag{4}$$

Thus, the number of nodes above the real line equals the number of nodes below the real line. And the nodes on the real line are divided in half arbitrarily. □

References

[1] F. T. Leighton, *Introduction to Parallel Algorithms and Architectures: Arrays - Trees - Hypercubes.* Morgan Kaufmann, 1992.