| 6.895 Theory of Parallel Systems | Due: Thursday, November 27th |
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# Problems 12–15

**Problem 12.** Denote the butterfly network running from major cycles to minor cycles by $W$, let its "transpose" $W^{\mathsf{T}}$ be the butterfly network running from minor cycles to major cycles. Thus, the Beneš network is essentially the network $WW^{\mathsf{T}}$. We've seen that $WW^{\mathsf{T}}$ can route any permutation (offline) by node-disjoint paths.

**Problem 12.a.** Show that the network $WWWW$, formed by 4 butterfly networks connected in tandem, can route any permutation through node-disjoint paths. (*Hint:* Show that $WW$ can implement the ***bit-reverse*** permutation.)

**Problem 12.b.** (*Harder.*) Show that the network $WWW$ can route any permutation by node-disjoint paths.

**Problem 12.c.** (*Open research problem.*) Can the network $WW$ route any permutation by node-disjoint paths?

**Problem 13. Shuffling**

**Problem 13.a.** Show that $r$ perfect out-shuffles return an $N = 2^d$-card deck to its original order if and only if $2^r \equiv 1 \pmod{N-1}$.

**Problem 13.b.** Show that 8 perfect out-shuffles are necessary and sufficient to restore a 52-card deck to its original order.

**Problem 13.c.** How many perfect out-shuffles are necessary and sufficient to restore a 54-card deck to its original order?

**Problem 14. De Bruijn**

**Problem 14.a.** Construct a 32-bit de Bruijn sequence.

**Problem 14.b.** Prove that the bisection width of an $N$-node de Bruijn network is $O(N/\lg N)$. If you wish, you may use the fact that the bisection width of an $N$-node shuffle-exchange network is $O(N/\lg N)$.

**Problem 15. Kautz networks**
A $\boldsymbol{(d,k)}$ ***Kautz network*** is defined as follows:

- Each node is a word $\langle x_1, x_2, \ldots, x_k \rangle$ on the alphabet $\{0, 1, \ldots, d\}$, where $x_i \neq x_{i+1}$ for $i = 1, 2, \ldots, k-1$.
- For each node $\langle x_1, x_2, \ldots, x_k \rangle$ and all $z \in \{0, 1, \ldots, d\} - \{x_k\}$, an edge exists from $\langle x_1, x_2, \ldots, x_k \rangle$ to $\langle x_2, \ldots, x_k, z \rangle$.

**Problem 15.a.** Draw a $(2,3)$ Kautz network.

**Problem 15.b.** Prove that every node in a $(d,k)$ Kautz network has out-degree $d$ and in-degree $d$. Prove (careful: two cases) that the diameter is $k$.

**Problem 15.c.** How many nodes $N$ does a $(d,k)$ Kautz network contain in terms of $d$ and $k$? (Give a precise answer, not a bound.) Prove it.

**Problem 15.d.** Conjecture a good upper bound on the bisection width of a $(d,k)$ Kautz network. *Optional:* Prove your bound.

**Problem 15.e.** Some architects have recently become enamored with the idea of interconnecting the processors of a parallel computer with a $(d,k)$ Kautz network, typically with $d \geq 4$. List salient pros and cons.