6.890	Lecture 21	Nov. 18. 2014

3SUM: [Gajentaan & Overmars - CGTA 1995]
given n integers, do any 3 sum to 0?
(allowing same integer to be chosen > once)

- conjecture: no $O(n^{2-\varepsilon})$ algorithm
  "truly subquadratic"
- $O(n^2)$ randomized algorithm:
  - compute all pairwise sums
  - look in hash table of all negations
- $O(n^2)$ deterministic algorithm:
  - presort integers
  - for each target sum (negated integer):
    - advance left pointer right if too small
    - advance right pointer left if too big
- $O(n + u \lg u)$ via FFT if integers $\in [-u, u]$
- $O\left(n^2 / \left(\frac{\lg n}{\lg\lg n}\right)^2\right)$ randomized in word RAM
  [Baran, Demaine, Pătraşcu - Alg. 2008]
- $O\left(n^2 / \frac{\lg n}{(\lg\lg n)^2}\right)$ det.; $O\left(n^2 / \left(\frac{\lg n}{\lg\lg n}\right)^{2/3}\right)$ rand. in real RAM
  $O(n^{1.5}\sqrt{\lg n})$ in decision tree model
  [Grønlund & Pettie - FOCS 2014]

<u>k-SUM</u>: given n integers, do any k sum to 0?

- $O(n^{\lceil k/2 \rceil})$ randomized algorithm
- <u>conjecture</u>: no $O(n^{\lceil k/2 \rceil - \varepsilon})$ algorithm
- NP-complete for k an input ($\approx$ Partition)
- W[1]-hard w.r.t. k (but quadratic parameter blowup from Clique $\Rightarrow n^{o(\sqrt{k})}$ lower bound)
- ETH $\Rightarrow$ no $n^{o(k)}$ algorithm for k-SUM
  $\leq n^{0.99}$ [Pătraşcu & Williams - SODA 2010]

<u>3SUM-hard</u> = $O(n^{2-\varepsilon})$ algorithm $\Rightarrow$ one for 3SUM

<u>3SUM reduction</u> = $O(1)$-call reduction on $n' = O(n)$ running in $O(n^{2-\varepsilon})$ time

- A 3SUM-hard (e.g. 3SUM) $\Rightarrow$ B 3SUM-hard

<u>Base 3SUM-hard problems</u>: (all equivalent)

- 3SUM with $u = n^3$ via hashing [Pătraşcu - STOC 2010]
  $\approx$ [Baran, Demaine, Pătraşcu - Alg. 2008]
- <u>Distinct 3SUM</u>: $\exists$ 3 distinct integers summing to 0?
  - reduction from 3SUM: also check for doubled/tripled ints.
  - reverse reduction?? [Mikhail Rudoy, today]
- <u>3SUM'</u>: given sets A, B, C of n integers $\exists a \in A, b \in B, c \in C$ such that $a + b = c$?
  - reduction from 3SUM: $A = B = S$, $C = -S$ (or $a+b+c=0$)
  - also reduction in reverse direction [Gajentaan & Overmars - CGTA 1995]

- GeomBase: given n points in 2D with $y \in \{0, 1, 2\}$ $\exists$ nonhorizontal line hitting 3 points?
  - reduction from/to 3SUM:
    - $a \in A \leftrightarrow (a, 0)$
    - $b \in B \leftrightarrow (b, 2)$
    - $c \in C \leftrightarrow (c/2, 1)$

$$a + b = c \iff c/2 = \frac{a+b}{2}$$

[Gajentaan & Overmars – CGTA 1995]

More 3SUM-hard problems:
[Gajentaan & Overmars – CGTA 1995]
- also solvable in $O(n^2)$ time

3 points on a line: given n points in the plane, are any 3 collinear?
- reduction from Distinct 3SUM !
- $x \in S \rightarrow (x, x^3)$

Point on 3 lines: given n lines in the plane, do any 3 meet at a point?
- projective plane dual of 3 points on line:
  - $(a, b) \Leftrightarrow ax + by + 1 = 0$
    (lines $ax + by = 0$ passing through origin map to points @ infinity ~ avoid these)
  - preserves point/line incidence

d-D versions: $(d+1)$-SUM hard

Separator: given n segments, is there a line splitting them into 2 nonempty groups?
- reduction from GeomBase
- if allow half-infinite segments, can all be horizontal (Sep. 1)
- else horizontal & vertical segments (Sep. 2)

Strips cover box: does union of n strips cover a given axis-aligned rectangle?


parallel

- reduction from GeomBase
- start from Separator 1 reduction rotated 90°
- dualize: $(m, b) \to y = mx + b$
  - vertical segment $\to$ strip
  - half-infinite segment $\to$ half plane
- rectangle = bounding box of hexagonal hole in union of 6 half-planes
- restrict half planes to this rectangle $\to$ 6 more strips
- uncovered point in dual = line in primal not hitting any segments

Triangles cover triangle:

- reduction from previous problem
- convert box → triangle with $O(1)$ strips
- split strips into 2 large Δs

- can assume n triangles $\subseteq$ big triangle:
  - replace each triangle with intersection
  - triangulate resulting $O(1)$-gons

Hole in union: does union of n triangles have a hole?

- reduction from previous problem ($\subseteq$ version)
- add thin Δs covering edges of big Δ
- hole $\Leftrightarrow$ not covered
- reduction in reverse direction also possible

Triangle measure: area of union of n triangles

- reduction from Triangles cover triangle ($\subseteq$)
- area(union) = area(big Δ) $\Leftrightarrow$ covered

Point covering: is there a k-way intersection between n given half planes?

- reduction from Strips cover box
- strip → complement as 2 half planes
- rectangle → 4 half planes whose int. = rect.
- $k = n+4$ (outside n strips, inside rectangle)

Visibility between segments:
given n horizontal segments, is there a point on segment 1 that can see a point on segment 2 (unobstructed by segments)
- reduction from GeomBase like Separator 1

Visible triangle: given n horizontal triangles in 3D can a given point see a point on triangle 1?
- reduction from Triangles cover triangle (view from infinity)
- reduction in reverse direction too

Planar motion planning: can you move segment robot through horizontal & vertical segment obstacles?
↳ translate & rotate
- reduction from GeomBase (like Separator 1)

3D motion planning: can you translate vertical segment robot through horizontal Δ obstacles?
- reduction from Triangles cover triangle
- separate Δs slightly in z, in middle of cage
- goal: get from top half to bottom half of cage
- $O(n^2 \lg n)$ algorithm

Fixed-angle chains: [Soss, Erickson, Overmars 2002]
which edge-spin operations cause collisions in a given fixed-angle chain?

- reduction from 3SUM'
- subtract $2M$ from each $a \in A$ $\rightarrow A'$
- add $2M$ to each $c \in C$ $\rightarrow C'$
  M = max abs $(A \cup B \cup C)$
- best algorithm: $O(n^3)$ [Soss & Toussaint 2001]

Nonquadratic lower bounds: [Pătrașcu - STOC 2010]

- finding Δ of prescribed weight in a weighted graph in $O(E^{1.5-\varepsilon})$ time is 3SUM-hard (as hard as $O(n^{2-\varepsilon})$ for 3SUM)
- finding $|E|$ Δs in $O(E^{4/3-\varepsilon})$ time is 3SUM-hard

Conjectured cubic graph problems: (weighted)

Diameter: $\max_{v,w} \delta(v,w)$ in undirected graph

- conjecture: no $O(V^{3-\varepsilon})$-time algorithm
  $\hookrightarrow O(n^{3-\varepsilon})$
- no $(3/2-\varepsilon)$-approx. in $O(E^{2-\varepsilon})$ time, even unweighted, assuming Strong ETH [Roditty & Vassilevska Williams — T.ALG 2012]
- subcubic reduces to:
  $\hookrightarrow O(n^{3-\varepsilon})$

APSP (All-Pairs Shortest Paths): $\delta(v,w)\ \forall v,w$

- $O(V^3)$ via Floyd-Warshall algorithm (relax all edges $|V|$ times)
- conjecture: no $O(V^{3-\varepsilon})$-time algorithm
- APSP-hard = no $O(V^{3-\varepsilon})$ alg. assuming

Negative Δ: is there a 3-cycle of negative weight?

- APSP-hard ~ actually equivalent
- equivalent to listing $|V|^{0.99}$ negative Δs
- equivalent to testing Δ inequality

[Vassilevska Williams & Williams — FOCS 2010]

Radius: $\min_v \max_w \delta(v,w)$

Median: $\min_v \sum_w \delta(v,w)$

[Abboud, Grandoni, Vassilevska Williams — SODA 2015]

- APSP-hard ~ actually equivalent (directed or undirected)