0-player games (simulations)
- polynomial # moves $\Rightarrow P$
- polynomial space $\Rightarrow$ PSPACE
- infinite space $\Rightarrow$ undecidable

Conway's Game of Life: [Conway 1970]
- cellular automata
- live cell lives $\Leftrightarrow$ exactly 2 or 3 live neighbors
- dead cell becomes live $\Leftrightarrow$ exactly 3 live neighbors
- PSPACE-complete in finite board
  - Turing machine simulation [Paul Rendell 2000]
    (pushdown automaton with 2 stacks)
  - undecidable in infinite board (dead outside input)
  - growing Turing machine [Paul Rendell 2000]
  - 2-counter machine = Minsky machine = exponential slowdown!
    [Conway, Berlekamp, Guy – Winning Ways 1982]
- wire, terminator, turn, delay
- shift: many offsets cause glider destruction
- AND & OR gates
- kickback $\Rightarrow$ thinning $\Rightarrow$ crossover
- split & NOT (complicated)
- counter registers, test, create, push/pull
- precise glider positioning away from guns
- self-destruction via boomerangs
Deterministic Constraint Logic (DCL)
- edges can also be active or inactive
- vertex active if its active incoming edges' weight $\geq 2$
- in each round:
  - reverse inactive edges pointing to active vertices
  - reverse active edges pointing to inactive vertices
  - these are the new active edges

- PSPACE-complete even for planar AND/OR graphs
  - guarantee gadget inputs reverse at $t = 0 \mod 4$
  - quantifier gadgets use new "switch" & degree-2 vertices to control timing
  - CNF formula uses AND, OR, split gadgets which take inputs & return acknowledgments (fixes timing & "blow-back")
  - trick to guarantee first input of AND activates before second (if they both do)
- remove degree-2 vertices
  - edge -> 4-path & remove red-red vertices
  - remove blue-blue vertices
  - remove red-blue vertices (timing is OK)
- crossover gadget
Multiplayer games:
- typical question: given a game position, can next-player-to-move force a win?
- in worst case, other players collude against you, effectively acting as one player

2-player games:
- call players “white” & “black” (as in Chess, Go, …)
- polynomial # moves $\rightarrow \in \text{PSPACE}$

$\exists$ move : $A$ responses : $\exists$ move : $A$ responses : … 
(I followed the rules & I won) $\lor$ (you broke rules)

SAT games: [Schaefer – JCSS 1978]
- QSAT is a 2-player game: $G_{w}(\text{CNF})$
  - player 1 chooses $x_1$, player 2 chooses $x_2$, …
  - player 1 wins $\iff$ formula satisfied

- impartial games: (both players have same moves)
  - on turn, player sets any unassigned variable

- partizan games: (different moves for players)
  - white variables & black variables (50/50%)
  - on turn, player sets unassigned var. of same color

- default game: player 1 wins $\iff$ formula satisfied
- seek game: win if first to satisfy formula
- avoid game: lose if first to satisfy formula
- PSPACE-complete: $\Rightarrow$ 11-CNF
  - impartial game positive 11-SAT
  - impartial game positive 11-DNF SAT
  - partizan game CNF SAT
  - impartial/partizan avoid positive 2-DNF SAT
  - impartial/partizan seek positive 3-DNF SAT
  - impartial/partizan avoid positive CNF SAT
  - impartial/partizan seek positive CNF SAT

Kayles: ($\approx$ indep. set)  [Schaefer - JCSS 1978]
  - (impartial) node Kayles:
    - on turn, player adds node to independent set
    - lose if can't move
  - (partizan) bipartite node Kayles:
    - white vs. black nodes is the bipartition

Geography: (generalization of word game) ($\approx$ longest path)
  - given (directed) graph & start node for token
  - on turn, player moves token along (directed) edge
  - node geography: can't revisit nodes
    - directed PSPACE-complete  [Lichtenstein & Sipser 1980]
    - undirected $\in P$  [Fraenkel, Scheinerman, Ullman 1993]
  - edge geography: can't revisit edges
    - directed PSPACE-complete  [Schaefer - JCSS 1978]
    - undirected PSPACE-complete  [Fraenkel, Scheinerman, Ullman - TCS 1993]
Reversi/Othello:
- move = ○ ○ ○ ○ ○ ⇒ ○ ○ ○ ○ ○
- reverse in between 1 & 8 directions

- PSPACE-complete [Iwata & Kasai - TCS 1994]
- polynomial # moves: move consumes board
- reduction from directed node geography in bipartite max-degree-3 graph
  - rightward chains are threats by black:
    - black takes  \( \alpha \), then  \( \alpha' \), then corner, then all of bottom territory ⇒ win
- white wins if black can't move
- degree-2 vertices: \( \rightarrow \) & \( \rightarrow \)
- degree-3 vertices: \( \rightarrow \) & \( \rightarrow \)
  - if double visited then white or black wins
  - by black or white chooses
ASIDE:

**Bounded NCL**:
- NP-complete
  - each edge can be reversed only once
  - NP-complete for planar constraint graphs with AND, SPLIT, OR, CHOICE vertices
differ in initial edge orientations
- planar via crossover
- similar to proof of Constraint Graph Satisfaction

**Bounded 2-player Constraint Logic (2CL)**
- each edge is either white or black
- each edge can be reversed only once
- goal: each player has target edge & wins if they reverse it

- PSPACE-complete for planar constraint graphs with white AND, SPLIT, OR, CHOICE & VARIABLE vertex
- reduction from impartial game positive CNFSAT
- players take turns setting variables
- positive \(\Rightarrow\) white wants true, black wants false
- black can't win (edge irreversible)
- white wins \(\Leftrightarrow\) formula satisfied
- crossover gadget (only use of CHOICE)
- can make OR protected using free edge
  - no constraint at degree-1 end