6.890 | Lecture 8 | Sept. 30, 2014

Hamiltonian cycle/tour/circuit: (default)
= cycle visiting each vertex exactly once

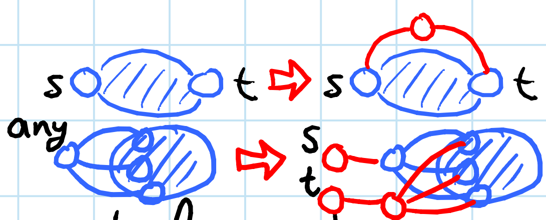
Hamiltonian path
= path visiting each vertex exactly once

History: Icosian Game [Sir William Rowan Hamilton 1857]
↳ Astronomer Royal of Ireland

Polynomial for (always YES)
- cubes of graphs [Karaganis 1968]
- planar 4-connected graphs [Tutte – Trans. AMS 1956]

NP-complete even: [Garey & Johnson book]

(P) – given start & end vertices
  – reduction to Ham. cycle:
  – reduction from Ham. cycle:



– for planar 3-regular 3-connected graphs with min. face degree 5 [Garey, Johnson, Tarjan 1976]
– for bipartite graphs [Krishnamoorthy 1975]
(C) – for squares of graphs [Chvátal 1976]
(C) – Ham. cycle given a Ham. path [Papadimitriou & Steiglitz 1976]

NP-complete for planar directed max-degree-3 graphs

in+out

[Plesník – IPL 1979]

- reduction from 3SAT
- clause gadget
- XOR gadget
- crossover gadget

NP-complete for planar bipartite max-degree-3 graphs

[Itai, Papadimitriou, Szwarcfiter – SICOMP 1982]

- reduction from previous problem

Grid graph = vertices on square lattice
+ edges for all pairs at unit distance

[Itai, Papadimitriou, Szwarcfiter – SICOMP 1982]

- solid grid graphs: Hamiltonicity polynomial
  ↳ no holes [Umans & Lenhart – FOCS 1997]
- (with holes) Hamiltonicity NP-complete [Itai et al. 1982]
- reduction from previous problem
  - parity-preserving grid embedding (via 3× scale)
  - edge (& turn) gadget "tentacles"
  - vertex gadget
    - ∃ Ham. path from p_i to p_j visiting e_1, e_2, e_3, e_4
  - vertex-edge connections (parity dependent)

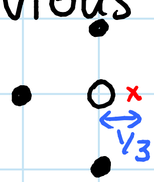
Euclidean TSP: NP-hard special case

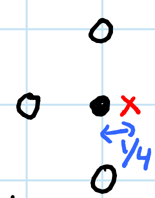
Platform games with coins & time limit [Forišek – FUN 2010]

Max-degree-3 grid graphs: [Papadimitriou & Vazirani 1984]

- similar reduction from planar bipartite max-deg-3 Ham.
- turn gadget → hole (but topologically same)
- vertex gadget = dumbbell
- vertex-edge connections:
  - degree-2 vertex: opposite ends of dumbbell
  - degree-3 vertex:
    - nonforced edges on opposite ends
    - forced edge on both ends via "fork"

Euclidean degree-3 MST: [Papadimitriou & Vazirani 1984]

- reduction from previous problem
- at white node: 

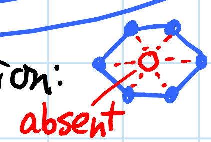
black node: 

- must connect added nodes to nearest neighbors

⇒ remainder is Ham. path

△ & hex grid graphs: [Arkin, Fekete, Islam, Meijer, Mitchell, Núñez-Rodríguez, Polishchuk, Rappaport, Xiao 2009]

- solid ⇒ △ & □ polynomial   ⬡ OPEN
- superthin ⇒ □ & ⬡ polynomial, △ NP-hard
  - ↳ all faces are holes/outside
- thin △ & □ & ⬡ NP-hard   [Demaine & Rudoy — arXiv:1706.10046]
  - ↳ every vertex on boundary
- max-deg. 4 △ NP-hard
- max-deg. 3 □ & ⬡ & △ NP-hard

⬡ → △ grid conversion:  [Aviv Adler & Mikhail Rudoy 2014]

- polygonal: △ polynomial, ⬡ NP-hard, □ NP-hard
  - ↳ no holes/outside face share a vertex (antisuperthin)

Settlers of Catan: [Klaus Teuber 1995]

- hex grid
- longest road → 2 Victory Points
- mate-in-1 is NP-hard [Demaine, van Eycke, McKay 2011]
  - opponents serve as obstacles
  - enough resources to buy all roads
  - get longest road ⟺ Hamiltonian
- "mate-in-0" is NP-hard [Demaine, van Eycke, McKay 2011]
  - have longest road ⟺ Hamiltonian

Slitherlink [Nikoli 1989]

- given grid of squares each blank or 0–4
- goal: find cycle on grid lines such that numbered squares have that many incident edges

- reduction from Planar Ham. cycle [Yato – IPSJ 2000]
  - optional vertex gadget
  - required vertex gadget
  - (non)edge connections
- reduction from Hamiltonicity in grid graphs

Hashiwokakero: [Nikoli 1990] "build bridges"

- given nodes with desired degrees
- goal: build orthogonal (multi)edges to connect nodes & satisfy degrees

- reduction from Hamiltonicity in grid graphs [Andersson – IPL 2009]
  - 1s for boundary
  - internal node = 2 + # boundaries

Milling: (NC milling) [Arkin, Fekete, Mitchell – CGTA 2000]

cut given region with given tool
using shortest path staying inside region

- NP-hard for grid polygon & unit □ tool
- reduction from Hamiltonicity in grid graphs:
  Minkowski sum of vertices with unit □
  ↳ set of all pairwise sums

Lawn mowing: (laser/waterjet/sign cutting)

path can go outside region

- NP-hard for grid polygon & unit □ tool
- same reduction: hurts length to leave region
- can even remove holes [Arkin, Fekete, Mitchell 2000]

3D printing: each layer is lawn mowing

Unit orthogonal segment intersection graphs:

- includes all grid graphs (rotated 45°)

$\Rightarrow$ Hamiltonicity NP-complete

[Arkin, Bender, Demaine, Fekete, Mitchell, Sethia – SICOMP 2005]

Minimum-turn milling: [Arkin et al. 2005]

- motivation: need to slow down for turns
- reduction from previous problem
  - segment → superthin rectangle
    → Minkowski sum with unit □
  - need 4 turns per segment
    + 1 turn per transition
  - 5n turns achievable $\Leftrightarrow$ Hamiltonian