Planar 3SAT:

- NP-hard special case of 3SAT
- variable-clause bipartite graph is planar
  - edge \((v_i, c_j)\) whenever \(v_i\) or \(\overline{v_i}\) is in \(c_j\)
- remains planar after connecting variables
  in a cycle: \(v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n \rightarrow v_1\)
- or after connecting variables & clauses
  in a cycle

+ remains planar if we require \(v_i\)'s positive
  connections separated from negative connections
  i.e. split \(v_i\) into \(v_{i+}\) and \(v_{i-}\)
  [Lichtenstein]
  positive connections  negative connections
  sometimes called “strongly planar”

+ remains planar if we require all positive
  connections on one side of cycle & negative
  connections on other side \(\Rightarrow\) monostone 3SAT
  [de Berg & Khosravi-Cocon 2010]

- also if \(\leq 2\) occurrences of each literal &
  \(\leq 3\) occ. of each var.
  [Brunner, Chung, Coulombe, Demaine, Gomez, Lynch - 2023]
- reductions from 3SAT
Planar rectilinear 3SAT: 
- variable = horizontal segment on x-axis
- clause = horizontal segment (off x-axis) + 3 vertical connections to variables
- no crossings/overlap (other than connections)

[Knuth & Raghunathan 1992]

Planar monotone rectilinear 3SAT: as above
+ monotone 3SAT: each clause all positive or all negative
+ positive clauses above x-axis
+ negative clauses below x-axis

[de Berg & Khosravi - COCOON 2010]

- reduction from planar rectilinear 3SAT

Careful:
- if all clauses on one side of variable cycle (above x-axis in planar rectilinear 3SAT)
  then CP via tree dynamic program

⇒ if clauses also connected in a path
  then CP (would force clauses on same side)
  (wanted this e.g. for Push-1/Nintendo)
Planar 1-in-3SAT: [Dyer & Frieze 1986]
- NP-hard special case of 1-in-3SAT
- variable-clause bipartite graph is planar
+ remains planar after connecting variables in a cycle: \( v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n \rightarrow v_1 \)
- OR after connecting variables & clauses in a cycle

Reduction from Planar 3SAT:
- clause gadget

Planar positive 1-in-E3SAT: no negations
also [Laroche 1993] \( \leftarrow \) [Mulzer & Rote - J. ACM 2008]
+ remains planar after connecting variables in a cycle: \( v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_n \rightarrow v_1 \)

Rectilinear:
- variable = horizontal segment on x axis
- clause = horizontal segment (off x axis)
  + 3 vertical connections to variables

Reduction from Planar 3SAT:
- equal & not-equal gadgets
- remove negations
- expand clauses (2 cases: \( u=0 \) or \( 1 \))
Careful: Planar NAE 3SAT is polynomial! [Moret-SIGACT News 1988]

Reduction to Planar Max Cut: 2-color vertices of planar graph to maximize red-blue edges

$4 \in P$ [Orlova & Dorfman 1972] [Hadlock-SICOMP 1975]
(in dual, red-blue edges are non-doubled edges in Chinese Postman problem)

- variable gadget / wire
- NAE clause

Planar X3C: [Dyer & Frieze 1986]

- bipartite graph of elements vs. 3-sets
  - is planar
  - reduction from planar 1-in-3SAT

Planar 3DM: [Dyer & Frieze 1986]

- special case where elements are 3-colored & each 3-set is trichromatic
- remains planar if elements connected in cycle
- reduction from planar 1-in-3SAT
Planar vertex cover: [Lichtenstein 1982]
- given a planar graph
- choose $k$ vertices to hit all edges
- reduction from planar 3SAT
  - variable gadget: even cycle
  - clause gadget: triangle
  - maximum degree 3

Planar (directed) Hamiltonian cycle: [Lichtenstein 1982]
- reduction from planar 3SAT
  - visit cycle through variables
  - variable gadget = ladder
  - clause gadget
  - can't jump var. $\Rightarrow$ clause $\Rightarrow$ other var.
- same reduction claimed for undirected

Shakashaka [Guten 2008; Nikoli 2012-]
- reduction from Planar 3SAT

Flattening fixed-angle chains: [Soss & Toussaint 2000; Demaine & Eisenstat 2011]
- reduction from Partition
- reduction from planar monotone rectilinear 3SAT