Recall: packing of \( n \) squares without rotation into a square is strongly NP-complete \([L2]\)

**Edge-unfolding polyhedra:** given a polyhedron, cut along edges to unfold flat without overlap

- not always possible \([\text{Biedl et al. & Bern et al. 1998}]\)
- strongly NP-hard \([\text{Abel & Demaine 2011}]\)
even for orthogonal polyhedra topologically sphere

**Reduction from Square Packing:**
- infrastructure: polyhedron with square with tower with squares & “atoms” on side
- “pipe” is super long but can move out
  \(\Rightarrow\) squares must pack inside base of tower
- atoms are universal: can turn left/right/straight in 2D unfolding & left/right/straight on tower surface
  \(\Rightarrow\) can connect & place squares as in any (slightly perturbed) packing, then exit via pipe
- lots of details e.g. shrink squares slightly to enable perturbation
Snake cube puzzle: AKA Cubra circa 1990
- given chain of unit cubes each with specified "turn angle" of 0 or 90° (elastic through centers)
- goal: fold it into larger cube (exactly)
- NP-hard

[Abel, Demaine, Demaine, Eisenstat, Lynch, Scharlau 2012]

Reduction from 3-Partition:
- infrastructure:
  - fill cube to leave $x \times y \times z$ box
  - fill box to leave "hub & slots" shape
  - each hub is $8 \times t \times$ huge
- $a_i$ gadget: 8$a_i$ must go in 1 hub
- 8 to avoid coming back to same 4×4×4 voxel
- connected together by zig-zag gadget
- zig-zag is universal:
  - 2×2×2 can turn/go straight
  - fill Hamiltonian shapes scaled 2×
  - 2×2×2 refinement makes any shape Hamiltonian
  - 4×4×4 refinement makes fillable by zig-zag
- parity issue: snake alternates in cell parity
- claim: can start & end at any faces of opposite parity
Disk packing: pack n given disks into given shape

- motivation: computational origami design
  (tree method – see Lang)

Reduction from 3-Partition:
- infrastructure:
  - build \( \frac{n}{3} \) symmetric \( \delta \) pockets
  - equilateral \( \Delta \): forced packing
  - square target: forced packing
    + repeated subdivision with forced packings
    + fill all other pockets by repeatedly
      adding maximal disks, until small enough
      (depth \( \approx \log n \))
  - triple gadget: (in symmetric pocket)
    - scale \( a_i \)'s & \( t \) so that \( t = 1 \)
    - shrink center disk by \( -\frac{1}{N} \)
    - shrink \( a_i \) disk by \( -\frac{1}{N^2} \), \( a_i \) big
    - grow it by \( +\frac{a_i}{N} \)
  - key property: disks fit \( \iff a_i + a_j + a_k \leq t \)
    (proof by geometry + Taylor series)
Clickomania: [Schuessler ~2000?]
- given rectangular grid of colored squares
- move = remove connected group of >1 square of the same color
- remaining squares fall within each column
- empty columns disappear

- polynomial for one row or column
- reduces to CFG parsing
- NP-hard for
- 2 columns & 5 colors
- 5 columns & 3 colors
- OPEN: 2 rows? 2 colors?

Reduction from 3-Partition:
- left column mostly checkerboard except middle & interspersed red squares to measure t's
- collapses \( \equiv \) red squares removed
- right column has \( a_i \) groups + red squares on top
- details: spacing out groups & reds while still getting alignment

[Schuessler, Demaine, Biedl, Fleischer, Jacobsen, Munro 2000]
Tetris: [Alexey Pazhitnov 1985]
- rectangular board
- tetromino blocks come one at a time
- 4 unit squares joined edge-to-edge
- can rotate block as it falls from sky
- filled lines disappear
- stack to sky => die

- perfect information version:
  - know entire sequence of pieces to come
  - initial board position given
- NP-complete to [Breukelaar, Demaine, Hohenberger, Hoogeboom, Koster, Liben-Nowell 2003]
  - survive
  - approximate # lines/Tetrises/time until death
  - up to a factor of \( n^{1-\varepsilon} \)

Reduction from 3-Partition: necessary: encoding in unary
- initial board = \( n/3 \) buckets of “depth” \( \epsilon \)
- \( a_i \) encoded as \( \square, \square, \square, \square, \square, \square \)
- claim: entire gadget must go in one bucket
- finale = \( \square, \square, \square, \square, \square, \square, \square, \square \)

Reachability: essentially all Tetris boards are reachable from initial state [Hoogeboom & Koster - IJIGS 2005]
OPEN:
- initially empty board
- $O(1)$ rows or columns
- restricted piece sets (e.g. 
- no last-minute slides
- 2-player: PSPACE-complete?
- online Tetris?

1-planarity: draw a given graph in the plane such that each edge crosses $\leq 1$ other

- NP-complete [Grigoriev & Bodlaender - Alg. 2007]

Reduction from 3-Partition:
- uncrossable edge gadget:
  \[ \Rightarrow \]
  (denoted by thick edge)
- double wheel gadget:
  - unique embedding
  - one for $A$
  - one for triples
  - separate triples with thick edges every $t$ hours around triples gadget
- $a_i$-gadget:
  A center triples center
GeoLoop & Ivan's Hinge puzzles: piano-hinged dissection

⇒ NP-complete from 3-Partition

Ruler folding:
- given carpenter's ruler with lengths $a_1, a_2, \ldots, a_n$
- goal: fold to fit in 1D box of length $L$

- weakly NP-complete \[\text{[Hopcroft, Joseph, Whitesides - 1985]}\]
- pseudopolynomial (like 2-Partition)

Reduction from (2-)Partition:
- idea: Partition solvable $\iff$ can assign signs to $a_i$'s such that $\sum_i a_i = 0$
- folding flips sign; unfolding leaves sign
⇒ can fold ends together $\iff$ Partition solvable

- construction: $2B, B, a_1, a_2, \ldots, a_n, B, 2B$

⇒ $2B$'s will be aligned & fit inside length-$2B$ box
⇒ can fold ends together $\iff$ Partition solvable
Map folding (simple): given crease pattern, can it fold flat by sequence of simple folds?

- weakly NP-hard [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena - 2006]
  for orthogonal paper & orthogonal creases
  or square paper & 45° orthog. creases

Reduction from Partition:
- similar to Ruler Folding
- 2 vertical creases check y extent against frame
- horizontal creases done before or after check
- force square paper into orthogonal shape:

\[
\begin{array}{c}
\text{OPEN:} \\
\text{UPDATE:}
\end{array}
\begin{array}{c}
\text{strongly NP-hard?} \\
\text{pseudopolynomial?} \\
\text{strongly NP-hard} \quad [\text{Akitaya, Demaine, Ku - JIP 2017}]
\end{array}
\]