6.891 Approximation Algorithms	March 31, 2000

Problem Set 4

Due Date: April 14, 2000

1. Consider the following problem:

   Minimum multicut problem in trees

   - **Input:**
     - A tree $T = (V, E)$
     - k pairs of vertices $(s_i, t_i), i = 1 \ldots k$
     - Costs $c_e \geq 0$ for each edge $e \in E$
   - **Goal:** Find a minimum-cost set of edges F such that for all i, s_i and t_i are in different connected components of $G' = (V, E - F)$.

   Obviously we can formulate this as a hitting set problem, where for each pair of vertices (s_i, t_i) we have a set $T_i = \{e : e \text{ on unique path from } s_i \text{ to } t_i\}$ that must be hit. Suppose we root the tree at an arbitrary vertex r. Let $depth(v)$ be the number of edges on the path from v to r. Let $lca(s_i, t_i)$ be the vertex v on the path from s_i to t_i whose depth is minimum. Suppose we use the primal-dual method to solve this problem, where the violated set T_i we choose in each iteration is the one that maximizes $depth(lca(s_i, t_i))$.

   Prove that this gives a 2-approximation algorithm for the minimum multicut problem in trees.

2. The minimum-cost branching problem is as follows:

   - **Input**:
     - An directed graph $G = (V, A)$
     - A root vertex $r \in V$
     - Weights $w_{ij} \geq 0$ for all $(i, j) \in A$
   - **Goal**: Find a minimum-cost set of arcs $F \subseteq A$ such that for every $v \in V$, there is a directed path in F from r to v.

   (a) Formulate the minimum-cost branching problem as a hitting set problem.

   (b) Use the primal-dual method and the analysis given in class to give an optimal algorithm for this problem.

3. Consider the following problem:

Partition into k-paths

- **Input:**
  - An unweighted graph $G = (V, E)$
  - Edge costs $c_e \geq 0$ for all $e \in E$ that obey the triangle inequality (i.e. $c_{(u,v)} \leq c_{(u,w)} + c_{(w,v)}$ for all $u, v, w \in V$).
  - Parameter k such that $|V|$ is a multiple of k.
- **Goal:** Find a minimum-cost collection of paths of k edges each such that each vertex is on exactly one path.

Consider also the related problem:

Partition into $0($mod $k)$-trees

- **Input:**
  - An unweighted graph $G = (V, E)$
  - Edge costs $c_e \geq 0$ for all $e \in E$
  - Parameter k such that $|V|$ is a multiple of k.
- **Goal:** Find a minimum-cost collection of trees such that each tree has $0($mod $k)$ vertices, and each vertex is in exactly one tree.

(a) Given an α-approximation algortithm for the second problem, produce a $2\alpha\left(1 - \frac{1}{k}\right)$-approximation algorithm for the first.

(b) Use the primal-dual method to give a 2-approximation algorithm for the second problem.

(c) Give a $4\left(1 - \frac{1}{k}\right)$-approximation algorithm for the problem of partitioning a graph into *cycles* of length exactly k.

4. Consider the following problem:

Prize-collecting Steiner tree problem

- **Input:**
  - An undirected graph $G = (V, E)$
  - Edge costs $c_e \geq 0$ for all $e \in E$
  - Penalties $\pi_i \geq 0$ for all $i \in V$
  - A root vertex $r \in V$
- **Goal:** Find a tree T containing the root r that minimizes the sum of the costs of the edges in the tree and the penalties of the vertices not in the tree (that is, that minimizes $\sum_{e \in T} c_e + \sum_{i \notin V(T)} \pi_i$, where $V(T)$ is the set of vertices in tree T.)

(a) Formulate this problem as a hitting set problem.

(b) (Bonus problem) Using the primal-dual method, give a 2-approximation algorithm for this problem.