6.891 Approximation Algorithms March 10, 2000

Problem Set 3

Due Date: March 31, 2000

1. Consider the problem of scheduling n jobs on m identical machines, where p_j is the amount of processing time required for job j. The goal is to minimize the maximum completion time C_j over all jobs j. This problem is denote $P||C_{\max}$. It is not hard to show that a simple "list scheduling" algorithm, which takes the list of jobs, and schedules the next job in the list on the least loaded machine, is a 2-approximation algorithm. (See pages 21–22 from last years notes for this argument).

   Suppose we begin by sorting the list so that $p_1 \geq p_2 \geq \cdots \geq p_n$. Prove that this gives a $\frac{4}{3}$-approximation algorithm. (Extended hint: Suppose job k finishes last. What can we say if $p_k \leq OPT/3$? Now suppose $p_k > OPT/3$. Argue that the performance of the algorithm on jobs $J_1, J_2, \ldots, J_k$ will be no better than that on $J_1, J_2, \ldots, J_n$. Now argue that the algorithm does well in this case when $p_j > OPT/3$ for all jobs j.)

2. As with linear programs, semidefinite programs have duals. The dual of the MAX CUT SDP we looked at is

$$\begin{array}{lll} & \frac{1}{2}\sum_{i<j} w_{ij} + \frac{1}{4}\text{Min} & \sum_i \gamma_i \\ (D) & \text{subject to:} & W + diag(\gamma) \text{ symmetric, positive semidefinite,} \end{array}$$

   where the matrix W is the symmetric matrix of the edge weights w_{ij} and $diag(\gamma)$ is the matrix of zeroes with γ_i as the ith entry on the diagonal. Show that the value of any feasible solution for this dual is an upper bound on the cost of any cut.

3. Semidefinite programming can also be used to give improved approximation algorithms for the MAX SAT problem. First we start with the MAX 2SAT problem, in which every clause has at most two literals.

   (a) As in the case of MAX CUT, we'd like to express the MAX 2SAT problem as a "integer quadratic program" in which the only constraints are $y_i \in \{-1, 1\}$ and the objective function is quadratic in the y_i. Show that the MAX 2SAT problem can be expressed this way. (Hint: it may help to introduce a variable y_0 which indicates whether the value -1 or 1 is "TRUE").

   (b) Derive a .878-approximation algorithm for the MAX 2SAT problem.

(c) (Bonus problem) Use this .878-approximation algorithm for MAX 2SAT to derive a $(\frac{3}{4}+\epsilon)$-approximation algorithm for MAX SAT, for some $\epsilon > 0$. How large an ϵ can you get?

4. An important quantity in combinatorial optimization is called the *Lovász Theta Function*. The theta function is defined on undirected graphs $G = (V, E)$. One of its many definitions is given below as a semidefinite program:

$$\begin{array}{lll} \vartheta(\bar{G}) = & \text{Max} \ \sum_{i,j} b_{i,j} & \\ & \text{subject to:} & \\ & \quad \sum_i b_{ii} = 1 & \\ & \quad b_{ij} = 0 & \forall (i,j) \notin E \\ & \quad B = (b_{ij}) \succeq 0 & B \text{ symm} \end{array}$$

Lovász showed that $\omega(G) \leq \vartheta(\bar{G}) \leq \chi(G)$, where $\omega(G)$ is the size of the largest clique in G and $\chi(G)$ is the minimum number of colors needed to color G.

(a) Show that $\omega(G) \leq \vartheta(\bar{G})$.

(b) The following is a small variation in the vector program we used for graph coloring:

$$\begin{array}{lll} \text{Min} & \alpha & \\ \text{subject to:} & v_i \cdot v_j = \alpha & \forall (i,j) \in E \\ & v_i \cdot v_i = 1 & \\ & v_i \in \Re^n. & \end{array}$$

Its dual is

$$\begin{array}{lll} \text{Max} & -\sum_i u_i \cdot u_i & \\ \text{subject to:} & \sum_{i \neq j} u_i \cdot u_j \geq 1 & \\ & u_i \cdot u_j = 0 & (i,j) \notin E, i \neq j \\ & u_i \in \Re^n. & \end{array}$$

Show that the value of the dual (and thus the primal) is $1/(1 - \vartheta(\bar{G}))$.

5. A metric space (V, d) is a set of vertices V together with a distance metric d such that $d_{ij} \geq 0$ for all $i, j \in V$, $d_{ij} = d_{ji}$ for all $i, j \in V$, $d_{ij} = 0$ for $i, j \in V$ iff $i = j$, and $d_{ij} + d_{jk} \geq d_{ik}$ for all $i, j, k \in V$. A metric space is said to be ℓ_p-embeddable if, for some m, there exists points $u_i \in \Re^m$ for $i \in V$ such that

$$d_{ij} = \|u_i - u_j\|_p,$$

and a metric space (V, d) is ℓ_p-embeddable with distortion c if for some m there exists points $u_i \in \Re^m$ for $i \in V$ such that

$$d_{ij} \leq \|u_i - u_j\|_p \leq c \cdot d_{ij}.$$

Show that given a metric space (V, d), semidefinite programming can be used to find an ℓ_2-embedding with the minimum possible distortion. (Recall that for $x \in \Re^m$, $\|x\|_p = \sqrt[p]{x_1^p + x_2^p + \cdots + x_m^p}$).