6.891 Approximation Algorithms February 11, 2000

Problem Set 1

Due Date: February 25, 2000

1. The following problem arises in telecommunications networks, and is known as the *SONET ring loading problem.* The network consists of a cycle on n nodes, numbered 0 through $n-1$ clockwise around the cycle. Some set C of calls is given; each call is a pair (i,j) originating at node i and destined to node j. The call can be routed either clockwise or counterclockwise around the ring. The objective is to route the calls so as to minimize the total *load* on the network. The load L_i on link $(i, i+1 (\text{mod } n))$ is the number of calls routed through link $(i, i+1(\text{mod } n))$, and the total load is $\max_{1 \le i \le n} L_i$.

   Give a 2-approximation algorithm for the SONET ring loading problem.

2. Consider the weighted vertex cover problem.

   (a) Prove that any extreme point of the linear program

   $$\text{Min} \quad \sum_{i \in V} w_i x_i$$

   subject to:

   $$\begin{aligned} x_i + x_j &\ge 1 & \quad \forall (i,j) \in E \\ x_i &\ge 0 & \quad i \in V \end{aligned}$$

   has the property that $x_i \in \{0, \frac{1}{2}, 1\}$ for all i (Recall that an extreme point x is a feasible solution that cannot be expressed as $\lambda x^1 + (1-\lambda)x^2$ for $0 < \lambda < 1$ and feasible solutions x^1 and x^2 distinct from x.)

   (b) Give a $\frac{3}{2}$-approximation algorithm for the weighted vertex cover problem when the input graph is planar. You may use the fact that polynomial-time LP solvers return extreme points, and that there is a polynomial-time algorithm to 4-color any planar graph (i.e. the algorithm assigns each vertex one of four colors such that for any edge $(i,j) \in E$, vertices i and j have been assigned different colors).

3. In the weighted Set Cover problem, one finds a collection I of sets that minimizes $\sum_{j \in I} w_j$ such that

   $$|\bigcup_{j \in I} S_j| = |T|.$$

   Consider the partial cover problem, in which one finds a collection I of sets that minimizes $\sum_{j \in I} w_j$ such that

   $$|\bigcup_{j \in I} S_j| \ge p|T|,$$

where $0 < p < 1$ is some constant.

(a) Give a polynomial-time algorithm to find a solution to the partial cover problem in which the value is no more than $c(p) \cdot OPT$, where $c(p)$ is a constant that depends on p, and OPT is the value of the optimal solution to the set cover problem.

(b) (Bonus problem) Give an $f(p)$-approximation algorithm for the partial cover problem, such that f is nondecreasing in p and $f(1) \leq H_{|T|}$.

4. In the Karmarkar-Karp algorithm, we used *geometric grouping* in each iteration of the algorithm to create an instance I' from I. Consider the following grouping scheme: for $i = 0, \ldots, \lceil \log_2 SIZE(I) \rceil$, create a group G_i such that all pieces from I of size $(2^{-(i+1)}, 2^{-i}]$ are placed in group G_i. In each group G_i, create subgroups $G_{i,1}, G_{i,2}, \ldots, G_{i,k_i}$ by arranging the pieces in G_i from largest to smallest and putting the $4 \cdot 2^i$ largest in the first subgroup, the next $4 \cdot 2^i$ in the next subgroup, and so on. Now create instance I' from I in the following manner: for each $i = 0, \ldots, \lceil \log_2 SIZE(I) \rceil$, discard subgroups $G_{i,1}$ and G_{i,k_i} (i.e., the first and last subgroups of G_i) and for $j = 2, \ldots, k_i - 1$ round each piece of subgroup $G_{i,j}$ to the size of the largest piece in $G_{i,j}$.

   Prove that by using this grouping scheme within the Karmarkar-Karp algorithm, we obtain an algorithm KK' such that $KK'(I) \leq OPT(I) + O(\log^2(OPT(I)))$ for all instances I.

5. Consider the following problem:

   Directed Steiner Tree

   - **Input:**
     - A directed graph $G = (V, E)$
     - Costs $c_{ij} \geq 0$ for arcs $(i, j) \in E$
     - A root vertex $r \in V$
     - A set of terminal $T \subseteq V$
   - **Goal:** Find a minimum-cost tree such that for each $i \in T$ there exsts a directed path from r to i.

   Prove that for some constant c there can be no $c \log |T|$-approximation algorithm for the Directed Steiner Tree problem unless $P = NP$.

   (Research problem: Give a $c \log^k |T|$-approximation algorithm for the Directed Steiner Tree problem for some constants c and k).