Lagrange Multipliers

In general, to find the extrema of a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ one must solve the system of equations:

$$\frac{\partial f}{\partial x_i}(\vec{x}) = 0$$

or equivalently:

$$\vec{\nabla} f = \vec{0}.$$

The method of lagrange multipliers is used to find extrema of a function under some constraints. In other words, one can find the extrema of a function

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

while satisfying some m constraints

$$g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$$g_j(\vec{x}) = 0.$$

We can approach this problem by defining a function

$$F : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}$$

$$F(\vec{x}, \vec{\lambda}) = f(\vec{x}) - \sum_j \lambda_j g_j(\vec{x}).$$

So, when we find the extrema of F subject to our constraints, we are solving our original problem. However, notice that in finding the extrema of F in $\mathbb{R}^{n+m}$ using the gradient method we produce the following equations:

$$\frac{\partial f}{\partial x_i}(\vec{x}) = \sum_j \lambda_j \frac{\partial g_j}{\partial x_i}(\vec{x})$$

$$g_j(\vec{x}) = 0$$

Solving these equations will therefore give extrema of f subject to the constraints g_j. The λ_js represent "forces of constraint."

Example: Find the dimensions of the largest rectangle with perimeter p.

Say the rectangle has dimensions $x \times y$. We are trying to maximize the area,

$$f(x, y) = x \cdot y,$$

subject to the constraint

$$g(x, y) = 2x + 2y - p = 0.$$

We must solve the system of equations:

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g &= 0 \end{aligned}$$

which becomes (after substitution):

$$\begin{aligned} y &= \lambda \cdot 2 \\ x &= \lambda \cdot 2 \\ 2x + 2y &= p. \end{aligned}$$

Substituting the first two equations into the third equation, we have:

$$\begin{aligned} 4\lambda &= p \\ \lambda &= p/4 \\ \Rightarrow x &= p/2 \\ y &= p/2. \end{aligned}$$

Thus, the largest rectangle with a given perimeter is the square with that perimeter.

Example: Find the point on the unit circle closest to the point (3,1).

Say the point has coordinates (x, y). We are trying to minimize the distance:

$$f(x, y) = (x - 3)^2 + (y - 1)^2,$$

subject to the constraint

$$g(x, y) = x^2 + y^2 - 1 = 0.$$

We must solve the system of equations:

$$\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ g &= 0 \end{aligned}$$

which becomes (after substitution):

$$\begin{aligned} 2(x - 3) &= \lambda \cdot 2x \\ 2(y - 1) &= \lambda \cdot 2y \\ x^2 + y^2 &= 1. \end{aligned}$$

We have:

$$\begin{aligned} x(1 - \lambda) &= 3 \\ y(1 - \lambda) &= 1 \\ x^2(1 - \lambda)^2 &= 9 \\ y^2(1 - \lambda)^2 &= 1 \\ x^2(1 - \lambda)^2 + y^2(1 - \lambda)^2 &= (1 - \lambda)^2 \\ 10 &= (1 - \lambda)^2 \\ 1 - \lambda &= \sqrt{10} \\ \lambda &= 1 - \sqrt{10} \\ x &= \frac{3}{\sqrt{10}} \\ y &= \frac{1}{\sqrt{10}}. \end{aligned}$$

Therefore, the point on the unit circle closest to (3,1) is $(3/\sqrt{10}, 1/\sqrt{10})$.