6.890: Algorithmic Lower Bounds — Prof. Erik Demaine
Fall 2014 — TAs: Sarah Eisenstat, Jayson Lynch

Problem Set 2

Due: Wednesday, October 8th, 2014

Problem 1. The R&D department of IncompeTech, Inc. has been looking into light-based circuits. In their years of research, they've developed a single type of gate. The gate takes two inputs — each either red light, green light, or blue light — and outputs a single beam of light whose color is determined by the following table:

input 2 \ input 1	***R***	***G***	***B***
R	G	R	B
G	R	B	R
B	B	R	G

IncompeTech's circuits take the form of a directed acyclic graph in which every node has in-degree 0 or in-degree 2. There are two types of nodes with in-degree 0: the inputs $x_1, \ldots, x_n$ and the constants R, G, and B. The nodes with in-degree 2 represent instances of the gate shown above. One such node produces the output (R, G, or B) of the overall circuit.

The R&D department of IncompeTech has designed several circuits which are supposed to be able to output the color red for an appropriate (unknown) selection of inputs, but the QA department hasn't figured out how to test for this. Can this property be checked in polynomial time, or is IncompeTech struggling with an NP-hard problem?

Problem 2. For each of the following planar edge-coloring problems, either show that the problem is NP-hard, or show that there exists a polynomial-time algorithm for the problem (e.g., by reducing to shortest paths, minimum spanning tree, matching, network flow, etc.).

(a) Given a 3-regular planar undirected multigraph (allowing parallel edges and self-loops), color each edge either red or blue such that, at each vertex, the multiset of colors of the incident edges is $\{R, B, B\}$.

(b) Given a 3-regular planar undirected multigraph (allowing parallel edges and self-loops), color each edge either red or blue such that, at each vertex, the multiset of colors of the incident edges is either $\{R, B, B\}$, $\{R, R, R\}$, or $\{B, B, B\}$.

(c) Suppose that you are given an planar undirected multigraph (allowing parallel edges and self-loops) such that each vertex has degree 3 or degree 6. Color each edge either red or blue such that, at each vertex, the multiset of colors of the incident edges is either $\{R, R, B, B, B, B\}$, $\{B, B, B\}$, or $\{R, R, R\}$.

Problem 3. Suppose that you are given an arbitrary planar graph, and a subset $Q \subseteq V$ of the vertices in the graph labeled "exactly 3 blue". Let $N(v)$ denote the set of neighbors of a vertex v. We want to color each vertex in the graph either red or blue such that, if we examine a vertex v labeled "exactly 3 blue", the number of blue nodes among $\{v\} \cup N(V)$ is exactly 3. (Unlabeled vertices are unconstrained.) Can this problem be solved in polynomial time, or is it NP-hard?

Problem 4. *Graduating is Hard, or: Why Does 6.890 Conflict With So Many Things?*
Students at the Miskatonic Institute of Technology sometimes find it hard to figure out whether they will graduate on time. The Institute has a set $\mathcal{C}$ of classes it offers, though not all classes are taught every semester. To graduate with a specific major, you must complete all of the courses required for that major. The courses you can take may also have additional restrictions, such as conflicts or prerequisites, described below.

(a) The major in Computational Metaphysics simply requires completing a specific set of required courses. Each course $c \in \mathcal{C}$ is offered in a certain set of semesters S_c. In each semester i, certain pairs of classes conflict, as given by an undirected graph $(\mathcal{C}, X_i)$. In each semester, you can take any set of classes that are offered during that semester and have no pairwise conflicts. Prove that it is NP-hard to decide whether you can graduate with a major in Computational Metaphysics.

(b) You notice that none of the classes required for a major in Eldritch Lore ever have conflicts with each other. However, due to the nature of the material, students are restricted to taking no more than four courses in any given semester. As before, each course $c \in \mathcal{C}$ is offered in a set of semesters S_c. In addition, each course $c \in \mathcal{C}$ has a prerequisite list $P_c \subseteq \mathcal{C}$. A student must have taken all of the prerequisites in P_c before they can take c. The graph of prerequisites form a (potentially disconnected) directed acyclic graph over the courses. Each course $c \in \mathcal{C}$ also has a list of courses R_c which can be used as substitutions for c, which fulfill all the prerequisite and major requirements that c does. Prove that it is NP-hard to decide whether you can graduate with a major in Eldritch Lore.

Problem 5. An ε-imperfect 2-coloring is a 2-coloring of an undirected graph in which no more than an ε fraction of the edges have endpoints of the same color. Show that it is NP-complete to find an ε-imperfect 2-coloring for some constant ε strictly between 0 and 1 (i.e., ε cannot have any dependence on the input but can be whatever constant you want other than 0 or 1).