6.890: ALGORITHMIC LOWER BOUNDS — Prof. Erik Demaine
Fall 2014 — TAs: Sarah Eisenstat, Jayson Lynch

Problem Set 1

Due: Monday, September 22nd, 2014

Problem 1. For each of the following problems, either show that the problem is in P by giving a polynomial-time algorithm (e.g., by reducing to shortest paths, network flow, matching, or minimum spanning tree); or show that the problem is NP-hard by reducing from 3-Partition, 3-Dimensional Matching, or Numerical 3-Dimensional Matching.

(a) Given a multiset of non-negative integers $A = \{a_1, \ldots, a_{2n}\}$ that sum to tn, find a partition of A into n groups $S_1, \ldots, S_n$ of size 2 such that each group sums to t.

(b) Given a multiset of non-negative integers $A = \{a_1, \ldots, a_{2n}\}$ that sum to tn, find a partition of A into n groups $S_1, \ldots, S_n$ of any size such that each group sums to t.

(c) Given a multiset of non-negative integers $A = \{a_1, \ldots, a_{2n}\}$ and a sequence of target numbers $\langle t_1, \ldots, t_n \rangle$, find a partition of A into n groups $S_1, \ldots, S_n$ of size 2 such that for each $i \in \{1, \ldots, n\}$, the sum of the elements in S_i is t_i.

Problem 2. Give a direct reduction from 3-Partition to Partition. (*Hint:* First reduce directly from 3-Partition to Subset-Sum, then modify the proof to work with Partition.)

Problem 3. Suppose you are given a weighted connected undirected graph $G = (V, E, w)$ satisfying the triangle inequality—that is, for any three vertices $x, y, z \in V$ connected in a triangle $(x, y), (y, z), (x, z) \in E$, we have $w(x, z) \leq w(x, y) + w(y, z)$. Your goal is to assign each node one of k colors. Define the *total weight* of a color be the sum of all of the distances between pairs of nodes of that color; where distance is is the weight of the minimum weight path between the nodes Show that it is NP-complete to find a color assignment in which the total weight of each color is less than t.

Problem 4. For each of the following problems, either show that it can be solved in polynomial time, or prove that the problem is NP-hard.

(a) You are trying to solve a $\sqrt{n} \times \sqrt{n}$ (unsigned) square edge-matching puzzle, which originally had n pieces. Unfortunately, you've managed to misplace $2/3$ of the puzzle pieces, leaving you with only $n/3$ pieces. A *configuration* of such a "partial" puzzle is a mapping of the remaining pieces onto the original $\sqrt{n} \times \sqrt{n}$ lattice; a configuration is *valid* if any two remaining pieces mapped to adjacent places match at their touching edges. How hard is it to solve (find a valid configuration of) the puzzle now?

(b) Several weeks later, while digging through the attic, you unearth another $1/3$ of the puzzle pieces, bringing you up to a total of $2n/3$ pieces of the original $\sqrt{n} \times \sqrt{n}$ puzzle. How hard is it to solve the puzzle now?