6.890	Lecture 5	Sept. 18, 2014

NP-hardness reductions from 3SAT / 1-in-3SAT / NAE 3SAT:

<u>Nintendo games:</u>

- Super Mario Bros. & World
  - glitches
- Legend of Zelda
  - push-once blocks
  - hookshot (A Link to the Past)
- Metroid
- Donkey Kong Country
- Pokémon
  - weak trainer: player always wins
  - strong trainer: player always loses

Phutball (Philosopher's Football) [Conway]

- one white stone (ball), many black stones (men)
- move = place new black stone
  OR "kick the ball" by jumping horizontally or vertically over black stones, immediately removing men, & repeat
- goal: reach opponent's side with ball

- PSPACE-hard [Dereniowski 2009]
- $\boxed{\text{OPEN}}$: EXPTIME-complete?
- NP-complete to decide mate-in-1: can you win in one move? (kick)
  - reduction from 3SAT [Demaine, Demaine, Eppstein 2000]

Checkers:

- EXPTIME-complete [Fraenkel, Garey, Johnson, Schaefer, Yesha 1978]
- mate-in-1 is polynomial
  - jumps preserve x & y parity
  - ⇒ Euler path problem

Cryptarithms / alphametics [Madachi 1979]

- given formula $x+y=z$ with each number written in base b & encoded with "letters" by unknown bijection between $\{0, 1, \dots, b-1\}$ & letters
- goal: feasible? / recover bijection
- strongly NP-complete [Eppstein 1987]

Reduction from 3SAT:

- variable gadget:
  - $b_i = 2a_i$
  - $v_i = 2b_i + C$ $\quad C = \text{carry}(y_i + y_i) \in \{0, 1\}$
    $= 4a_i + C \equiv C \pmod 4$
  - $d_i = 2c_i + C$
  - $e_i = d_i + 1 + C$
    $= 2c_i + 1 + 2C$
  - $\overline{v_i} = d_i + e_i$
    $= 4c_i + 1 + 3C$
    $\equiv 3C + 1 \equiv 1 - C \pmod 4$
- clause gadget:
  - $g_i = 2f_i$
  - $h_i = 2g_i + \{0, 1\}$
    $= 4f_i + \{0, 1\}$
  - $t_i = h_i + 1 + \{0, 1\}$
    $= 4f_i + 1 + \{0, 1, 2\}$
    $= 4f_i + \{1, 2, 3\}$
  - $v_a + v_b + v_c = t_i \equiv \{1, 2, 3\} \pmod 4$

Simplified reduction from 1-in-3SAT:

- variable gadget: just v_i, no $\overline{v_i}$ (monotone)
- clause gadget:
  - $g_i = 2f_i$
  - $h_i = 2g_i$
    $= 4f_i$
  - $t_i = h_i + 1$
    $= 4f_i + 1$
  - $v_a + v_b + v_c = t_i$
    $= 4f_i + 1 \equiv 1 \pmod 4$

3SAT solvable ⇒ cryptarithm solvable:

- distinguish $a_i, b_i, c_i, d_i, \ldots$ by value mod 128
- e.g. $v_i \equiv 8 \pmod{128}$ if true
  $\equiv 9 \pmod{128}$ if false
  $a_i \equiv \{2, 34, 66, 98\} \pmod{128}$
  ⋮
- set $\lfloor v_i / 128 \rfloor$ & $\lfloor \overline{v_i} / 128 \rfloor \in [0, (2n)^3]$
  such that distinct sums of triples
  [Bose & Chowla 1959]
- easy proof of polynomial range: (based on fusion trees)
  - if $<i$ set by induction, v_i must avoid
    $v_j + v_k + v_\ell - v_m - v_p$ ~ $<(2n)^5$ choices
  - ⇒ $(2n)^5$ suffices

⇒ strongly NP-hard

Origami: flat-foldable crease patterns

↳ each crease folded $\pm 180°$

↳ graph drawn in square with straight edges

↳ stacking order of overlapping faces

- strongly NP-hard [Bern & Hayes 1996]

Reduction from NAE 3SAT:

- wire = pleat
- NAE clause $\approx$ triangular twist
- splitter/negation (don't need negation)
- crossover

Vertex-disjoint paths:

- in a graph [Lynch 1975]
- in a planar graph [Lynch 1975]
- in a rectangle with all spots filled [Adcock, Demaine, Demaine, O'Brien, Reidl, Sánchez Villaamil, Sullivan 2014]
  - use terminals as obstacles
  - neighboring terminal pairs can just connect OR fill some uncovered space
  - issue 1: must have even-parity fill regions
  - issue 2: clause path may be absent! → more parity trouble → add row
  - issue 3: gadget width is odd parity → won't connect to split → add column
  - issue 4: crossing true or false line gadgets

= zig-zag Numberlink [Loyd 1897; Nikoli]

- classic Numberlink also NP-complete [Kotsuma & Takenaga 2010]