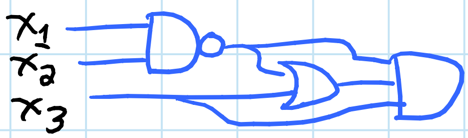
6.890	Lecture 4	Sept. 16, 2014

The most important NP-complete (logic) problem family!

SAT = Satisfiability: [Cook 1971; Levin 1973]
- given a Boolean formula (AND, OR, NOT) over n variables $x_1, x_2, \ldots, x_n$
- can you set x_i's to make formula true?

Circuit SAT: formula expressed as acyclic circuit of gates (allows re-use)



CNF SAT: formula = AND of clauses [Cook 1971]

Conjunctive Normal Form

clause = OR of literals

literal $\in \{x_i, \text{NOT } x_i\}$
- can view as bipartite graph: variables vs. clauses, positive/negative edges

3SAT: clause = OR of 3 literals [Cook 1971]

i.e. clause degrees = 3 (but allow repeats)

3SAT-3: each variable occurs in ≤ 3 clauses

$\neg x_1 \vee x_2$
$\neg x_2 \vee x_3$
$\vdots$
$\neg x_k \vee x_1$

- E3SAT-4 but E3SAT-3 $\in$ P [Tovey — DAM 1984]

exactly 3 distinct literals per clause

Monotone 3SAT: [Gold — I&C 1978] also -3

each clause all positive or all negative

Beware polynomial-time variants!

2SAT: clause = OR of 2 literals
- polynomial
- x OR y $\equiv$ NOT $x \Rightarrow y$ ($\equiv$ NOT $y \Rightarrow x$)
- guess x_i, follow all implication chains to check OK

BUT...

Max 2SAT: set variables to maximize # true clauses
- NP-complete [Garey, Johnson, Stockmeyer 1976]

Horn SAT: each clause has ≤ 1 positive literal
- NOT x OR NOT y OR NOT z OR w

$\equiv$ NOT (x AND y AND z) w

$\equiv$ (x AND y AND z) $\Rightarrow w$ [Horn 1951]

$\Rightarrow$ polynomial like 2SAT

Dual-Horn SAT: each clause has ≤ 1 negative literal

↳ "weakly positive satisfiability" [Schaefer 1978]
- negate all variables → Horn SAT

$\Rightarrow$ polynomial

Renamable Horn SAT: more generally, can negate some variables globally to make formula Horn

$\Rightarrow$ polynomial [Chandru, Coullard, Hammer, Montañez, Sun — Annals Math. & AI 1990]

DNF SAT: formula = OR of clauses

clause = AND of literals

↓

Disjunctive Normal Form

$\Rightarrow$ satisfiable $\Leftrightarrow \geq 1$ clause $\Rightarrow$ polynomial

Alternative clauses for 3SAT:

1-in-3SAT = exactly-1 3SAT [Schaefer 1978]
- clause = exactly 1 of 3 literals is true
  (⇒ 2 false ~ TFF, FTF, FFT)

omitted by Schaefer
Positive 1-in-3SAT: no negations – all literals positive
sometimes called "monotone"
BUT...
Positive not-exactly-1 3SAT: [Schaefer 1978]
- clause = 0, 2, or 3 variables are true
  i.e. $x_i \Rightarrow (x_j \text{ OR } x_k)$ → Dual Horn
- also require x_1 = TRUE (else set all x_i = FALSE)
  & x_2 = FALSE (or allow $|\text{clause}| \leq 3$)
- polynomial

NAE 3SAT = not-all-equal 3SAT [Schaefer 1978]
- clause = 3 literals not all the same value
  (forbid FFF & TTT ⇒ 1 or 2 true, 2 or 1 false
  ~ whereas 3SAT forbids just FFF)
- nice symmetry between TRUE & FALSE

omitted by Schaefer
Positive NAE 3SAT: no negations – all literals positive

Schaefer's Dichotomy: [Schaefer – STOC 1978]

- formula = AND of clauses
- general clause = relation on variables
  - assume in CNF (unique if minimal) ← use Karnaugh maps
  - ⇒ AND of subclauses
- ⇒ SAT is polynomial if either:
  - setting all variables true or all variables false satisfies all relations
  - subclauses are all Horn or all Dual Horn
  - relations are all 2-CNF (subclause sizes ≤ 2)
  - every relation can be expressed as a system of linear equations over $\mathbb{Z}_2$:

"XOR SAT" $\left\{ \; x_i \underset{\hookrightarrow \text{XOR}}{\oplus} x_j \oplus x_k \oplus x_\ell = 0 \text{ or } 1 \right.$ ↪ Gaussian elimination

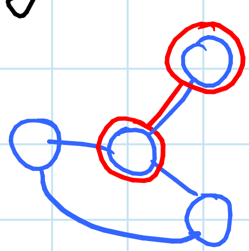
& otherwise, SAT is NP-complete!

Another hard version of SAT – seldom used?

2-colorable perfect matching: [Schaefer 1978]

- given a planar 3-regular graph
- 2-color the vertices such that every vertex has exactly 1 same-colored neighbor
- special case of 2-in-4 SAT

(planarity & 3-regular left as exercise)



Pushing blocks:

- 1×1 robot navigating grid of blocks
- goal: get robot from start to target

- Push-k: robot can push up to k blocks at once
- Push-*: infinite strength
- PushPush: blocks slide until they hit something
- PushPushPush: blocks slide other blocks (ice) in chain reaction, up to strength k
- Push⋯-F: some blocks are fixed
- Push⋯-X: robot path cannot self-intersect (tiles disappear after traversal)

- Sokoban = Push-1F but with goal of filling target squares with blocks

Push-*: reduction from 3SAT [Hoffmann 2000]

- variable: push right in x_i or $\overline{x_i}$ row
  → fill in row of connection gadget
- connection: 1 free cell per occurrence of literal
- bridge: move up & block off leftward path
- clause: need a free spot below to traverse
- applies to PushPush-* & PushPushPush-* too

PushPush-1 in 3D: reduction from 3SAT

[O'Rourke & Smith Problem Solving Group 1999]

(Push)Push-1 (in 2D): reduction from 3SAT

[Demaine, Demaine, O'Rourke 2000]

- clause gadget, block other, lock gadget
- NAND crossover: not both N→S & W→E
- unidir. crossover: optional N→S, then W→E
  - can leak N→S then W→S, which can enable revisiting old variables; luckily fork gadget forces single T/F assignment