6.890	Lecture 3	Sept. 11, 2014

Recall: packing of n squares without rotation into a square is strongly NP-complete [L2]

Edge-unfolding polyhedra: given a polyhedron, cut along edges to unfold flat without overlap

- not always possible [Biedl et al. & Bern et al. 1998]
- strongly NP-hard [Abel & Demaine 2011]
  even for orthogonal polyhedra topologically sphere

Reduction from Square Packing:
- infrastructure: polyhedron with square with tower with squares & "atoms" on side
- "pipe" is super long but can move out

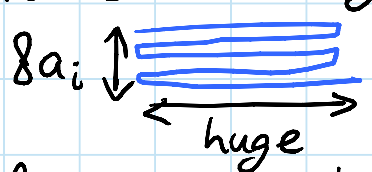
⇒ squares must pack inside base of tower
- atoms are universal: can turn left/right/straight in 2D unfolding & left/right/straight on tower surface

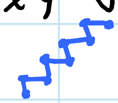
⇒ can connect & place squares as in any (slightly perturbed) packing, then exit via pipe
- lots of details e.g. shrink squares slightly to enable perturbation

Snake cube puzzle: AKA Cubra circa 1990

- given chain of unit cubes each with specified "turn angle" of 0 or 90° (elastic through centers)
- goal: fold it into larger cube (exactly)
- NP-hard
  [Abel, Demaine, Demaine, Eisenstat, Lynch, Schardl 2012]

Reduction from 3-Partition:

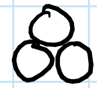
- infrastructure:
  - fill cube to leave x×y×z box
  - fill box to leave "hub & slots" shape
  - each hub is 8×t × huge
- a_i gadget: 

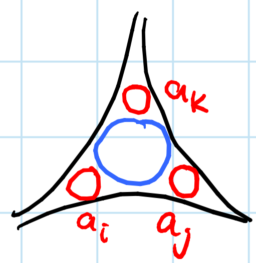
  must go in 1 hub
  - 8 to avoid coming back to same 4×4×4 voxel
- connected together by zig-zag gadget 
- zig-zag is universal:
  - 2×2×2 can turn/go straight
    ⇒ fill Hamiltonian shapes scaled 2×
  - 2×2×2 refinement makes any shape Hamilt.
  - ⇒ 4×4×4 refinement makes fillable by zig-zag
  - parity issue: snake alternates in cell parity
  - claim: can start & end at any faces of opposite parity

Disk packing: pack n given disks into given shape

- *motivation:* computational origami design (tree method — see Lang)
- strongly NP-hard [Demaine, Fekete, Lang — OSME 2010]

Reduction from 3-Partition:

- infrastructure:
  - build $n/3$ symmetric  pockets
  - equilateral Δ: forced packing
  - square target: forced packing
    + repeated subdivision with forced packings
    + fill all other pockets by repeatedly adding maximal disks, until small enough (depth $\approx \lg n$)
- triple gadget: (in symmetric pocket)
  - scale a_i's & t so that $t = 1$
  - shrink center disk by $-1/N$ ↳ big
  - shrink a_i disk by $-1/N^2$, grow it by $+a_i/N$
  - key property: disks fit $\Leftrightarrow a_i + a_j + a_k \le t$ (proof by geometry + Taylor series)



Clickomania: [Schuessler ~2000?]

- given rectangular grid of colored squares
- move = remove connected group of >1 square of the same color
  - remaining squares fall within each column
  - empty columns disappear

- polynomial for one row or column
  - reduces to CFG parsing

$S \to \Lambda \mid SS \mid c_i S c_i \mid c_i S c_i S c_i$

- NP-hard for
  - 2 columns & 5 colors
  - 5 columns & 3 colors

[Biedl, Demaine, Demaine, Fleischer, Jacobsen, Munro 2000]

- $\boxed{\text{OPEN}}$: 2 rows? 2 colors?

Reduction from 3-Partition: → necessary: encoding in unary

- left column mostly checkerboard except middle & interspersed red □s to measure t's
  - collapses ⇔ red □s removed
- right column has a_i groups + red squares on top ← scaled by B $= \frac{4}{3}n$
- details: spacing out groups & reds while still getting alignment

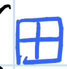
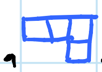
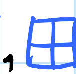
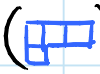
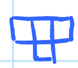
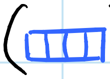
Tetris: [Alexey Pazhitnov 1985]

- retangular board
- tetromino blocks come one at a time
  - ↳ 4 unit squares joined edge-to-edge

- can rotate block as it falls from sky
- filled lines disappear
- stack to sky ⇒ die

- perfect information version:
  - know entire sequence of pieces to come
  - initial board position given
- NP-complete to [Breukelaar, Demaine, Hohenberger, Hoogeboom, Kosters, Liben-Nowell 2003]
  - survive
  - approximate # lines/Tetrises/time until death up to a factor of $n^{1-\varepsilon}$

Reduction from 3-Partition: → necessary: encoding in unary

- initial board = $n/3$ buckets of "depth" t
- a_i encoded as , (, , )^{a_i}, , 
  - claim: entire gadget must go in one bucket
- finale = ()$^{n/3}$, , ()$^{\frac{5}{4}t+4}$

Reachability: essentially all Tetris boards are reachable from initial state [Hoogeboom & Kosters – IJIGS 2005]

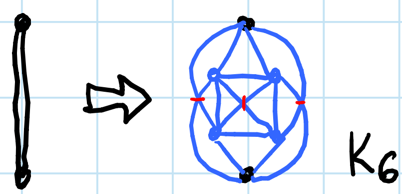
OPEN:
- initially empty board
- $O(1)$ rows or columns
- restricted piece sets (e.g. )
- no last-minute slides
- 2-player: PSPACE-complete?
- online Tetris?

1-planarity: draw a given graph in the plane such that each edge crosses ≤ 1 other [Ringel 1985]

- NP-complete [Grigoriev & Bodlaender — Alg. 2007]

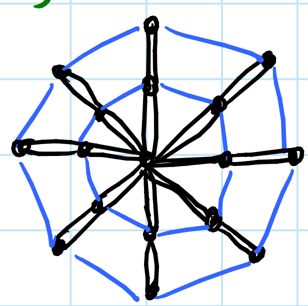
Reduction from 3-Partition:

- uncrossable edge gadget:

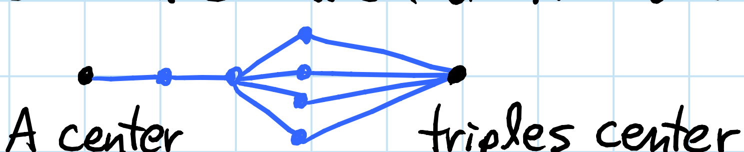


(denoted by thick edge)

- double wheel gadget:
  - unique embedding
  - one for summing
  - one for items
  - separate triples with thick edges every t hours around summing wheel & 3 hours around items wheel



- a_i-gadget:



GeoLoop & Ivan's Hinge puzzles: piano-hinged dissection
↳ NP-complete from 3-Partition

Ruler folding:
- given carpenter's ruler with lengths $a_1, a_2, \dots, a_n$
- goal: fold to fit in 1D box of length L

- weakly NP-complete [Hopcroft, Joseph, Whitesides — SICOMP 1985]
- pseudopolynomial (like 2-Partition)

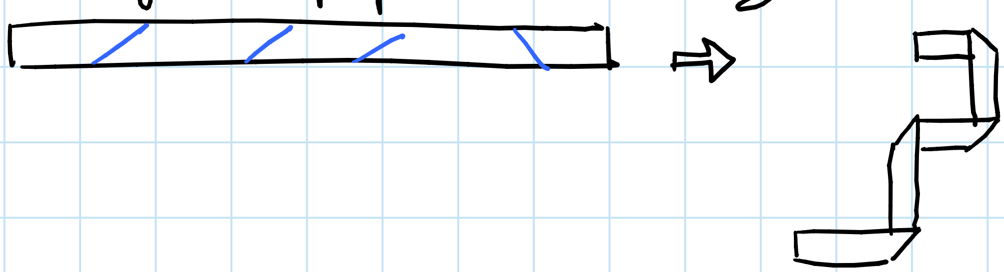
Reduction from (2-)Partition:
- idea: Partition solvable $\Leftrightarrow$ can assign signs to a_i's such that $\sum_i \pm a_i = \emptyset$
- folding flips sign; unfolding leaves sign
$\Rightarrow$ can fold ends together $\Leftrightarrow$ Partition solvable
- construction: $2B, B, a_1, a_2, \dots, a_n, B, 2B$
↳ $> \sum_i a_i$
$\Rightarrow$ $2B$'s will be aligned & fit inside length-$2B$ box
$\Leftrightarrow$ can fold ends together $\Leftrightarrow$ Partition solvable

Map folding (simple): given crease pattern, can it fold flat by sequence of simple folds?

- weakly NP-hard [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena – 2000]
  for orthogonal paper & orthogonal creases
  or square paper & 45°/orthog. creases

Reduction from Partition:
- similar to Ruler Folding
- 2 vertical creases check y extent against frame
- horizontal creases done before or after check
  if ruler folded ↲ ↳ if not

- force square paper into orthogonal shape:



~~OPEN~~: strongly NP-hard?
pseudopolynomial?

UPDATE: strongly NP-hard [Akitaya, Demaine, Ku – JIP 2017]