6.890	Lecture 1	Sept. 4, 2014

6.890: Algorithmic Lower Bounds / Fun with Hardness Proofs

"Hardness made Easy"

Prof. Erik Demaine
TAs: Sarah Eisenstat & Jayson Lynch
http://courses.csail.mit.edu/6.890/fall14/

What is this class?

- practical guide to proving computational problems are formally hard / intractable
- NOT a complexity course
  (but we will use/refer to needed results)
- (anti) algorithmic perspective

Why take this class?

- know your limits in algorithmic design
- master techniques for proving hardness — key problems, proof styles, gadgets
- cool connections between problems
- fun problems like Mario & Tetris
  (serious problems too)
- solve puzzles → publishable papers

Background: algorithms, asymptotics, combinatorics

- no complexity background needed
  (but also little overlap with a complexity class)

Requirements:

- fill out form (& join mailing list)
- attend lectures
- scribing 1-2 lectures
- ≈5 psets, 2-3 weeks each
- project & presentation
  - theory (attempt to solve/pose open problem)
  - survey (something not covered in class)
  - implement/visualize
  - Wikipedia
  - art (sculpture, etc. related to hardness)

Topics: (on handout/webpage)

- NP-completeness (3SAT, 3-partition, Hamiltonicity, geometry)
- PSPACE, EXPTIME, ...
- Games, Puzzles, & Computation (Constraint Logic, Sudoku, Nintendo, Tetris, Rush Hour, Chess, Go, ...)
- inapproximability (PCP, APX, Set Cover, ind. set, UGC, ...)
- fixed-parameter intractability (W, clique, ...)
- 3SUM (toward n^2)
- counting (#P) & uniqueness (ASP)
- economic game theory (PPAD)
- existential theory of reals, undecidability (if time)

Recommended texts: Games, Puzzles, & Computation [Hearn & Demaine]
Garey & Johnson

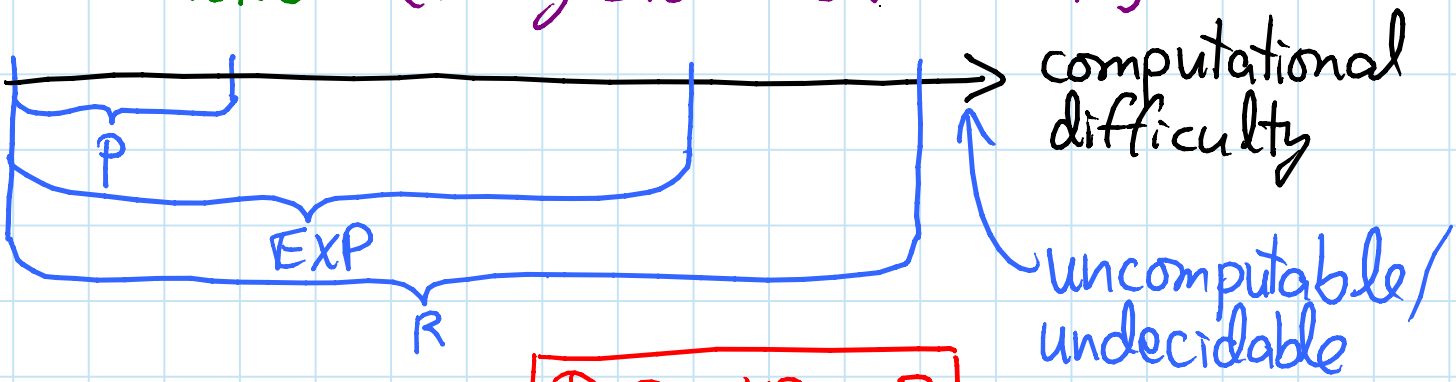
Intro to complexity:

P = {problems solvable in polynomial time} ($\to n^c$)

EXP = {problems solvable in exponential time} ($\to 2^{n^c}$)

R = {problems solvable in finite time}

↳ "recursive" [Turing 1936; Church 1941]



$P \subsetneq EXP \subsetneq R$

Examples:

- negative-weight cycle detection $\in P$
- $n \times n$ Chess $\in EXP$ but $\notin P$
  ↳ who wins from given board config.?
- Tetris $\in EXP$ but don't know whether $\in P$
  ↳ survive given pieces from given board
- halting problem $\notin R$
- "most" decision problems $\notin R$
  (# algorithms $\approx \mathbb{N}$; # dec. problems $\approx 2^{\mathbb{N}} = \mathbb{R}$)

NP = {decision problems solvable in poly. time via a "lucky" algorithm}

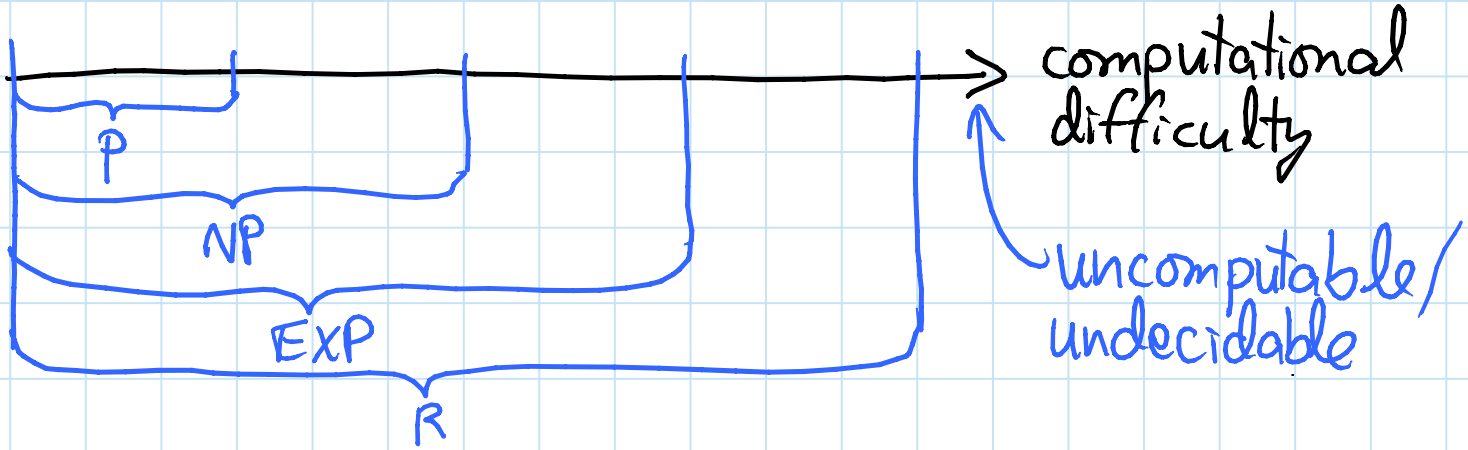
↗ answer ∈ {YES, NO}

↳ can make lucky guesses, always "right", without trying all options

- nondeterministic model: algorithm makes guesses & then says YES or NO
- guesses guaranteed to lead to YES outcome if possible (NO otherwise)

= {decision problems with solutions that can be "checked" in polynomial time}

- when answer = YES, can "prove" it & poly.-time algorithm can check proof



Example: Tetris ∈ NP

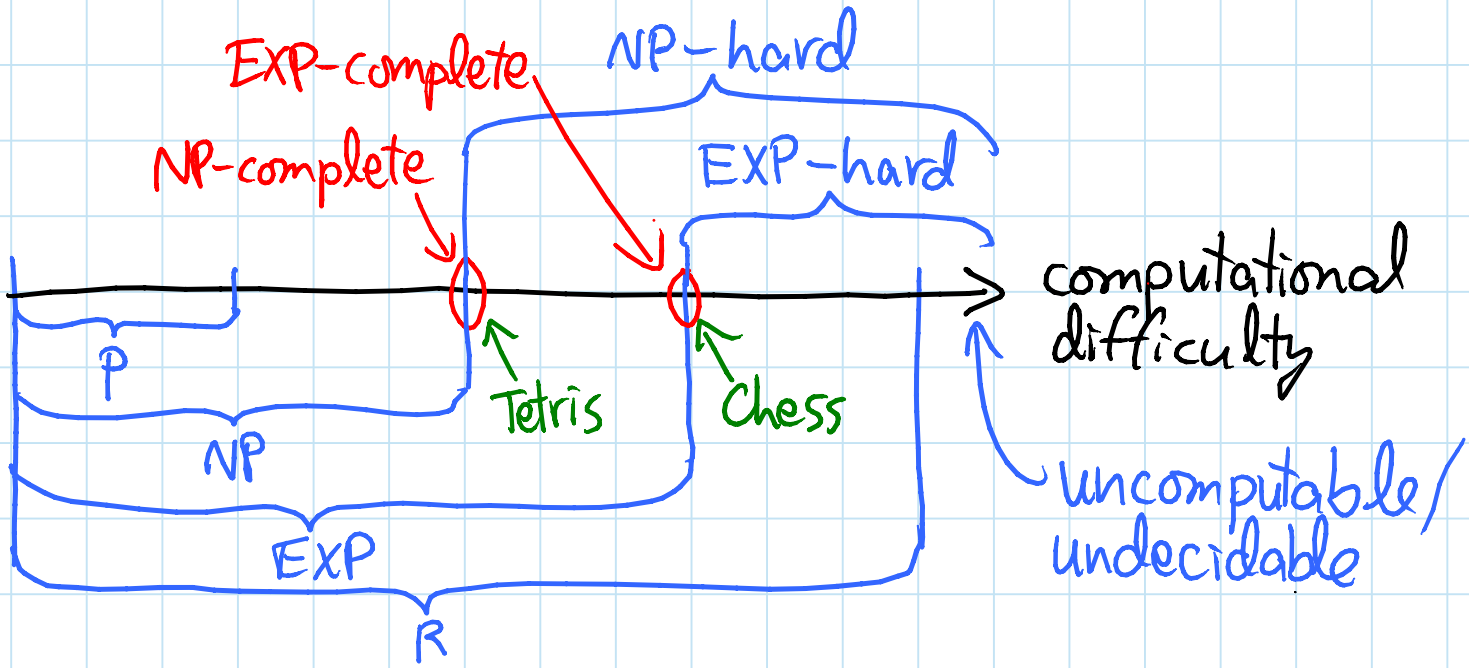
- nondeterministic alg: - guess each move
  - did I survive?
- proof of YES: list what moves to make (rules of Tetris are easy)

P≠NP: big conjecture (worth $1,000,000)
≈ can't engineer luck
≈ generating (proofs of) solutions can be harder than checking them

CoNP = negations (YES ↔ NO) of problems ∈ NP
= problems with good proofs of NO answer

X-hard = "as hard as" (defined later) every problem ∈ X (NP, EXP, etc.)
X-complete = X-hard ∩ X
sometimes "X-easy" = ∈X

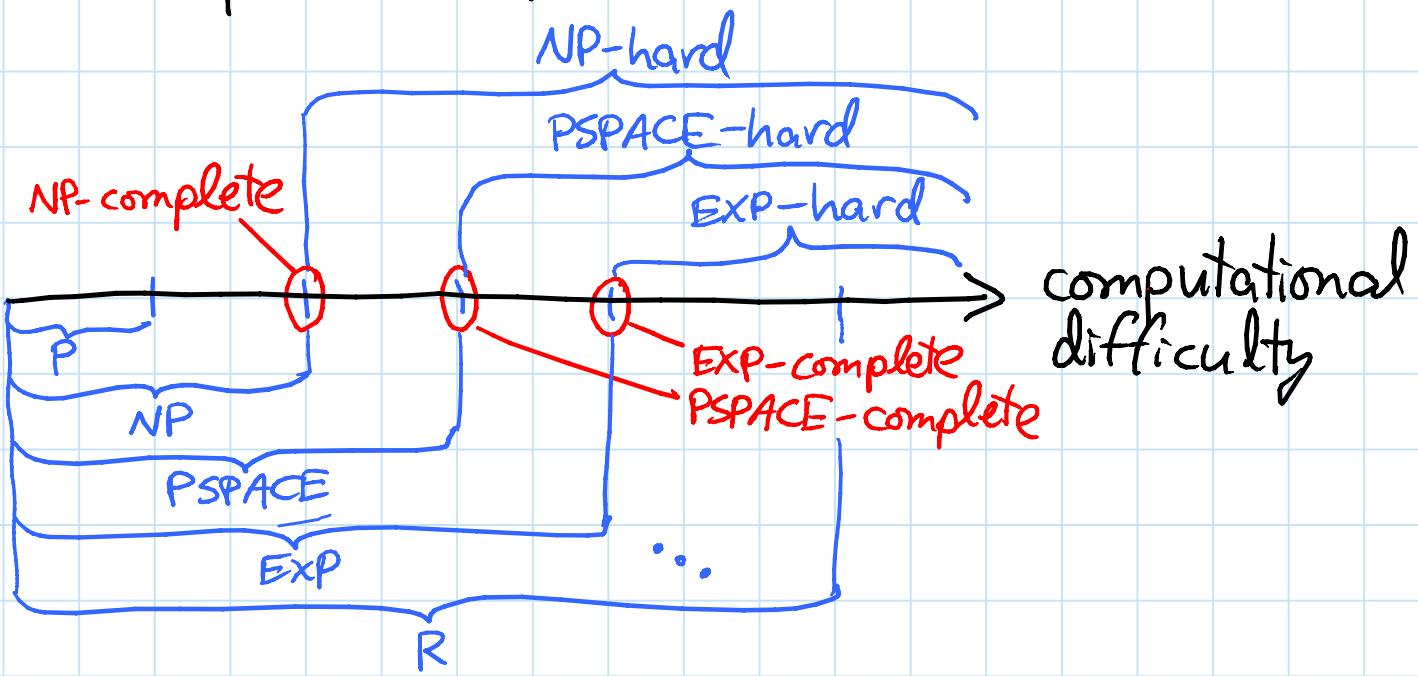
e.g. Tetris is NP-complete
[Breukelaar, Demaine, Hohenberger, Hoogeboom, Kosters, Liben-Nowell 2004]
⇒ if P ≠ NP, then Tetris ∈ NP \ P



e.g. Chess is **EXP-complete** ⇒ ∉ P
⇒ Chess ∈ EXP \ NP if NP ≠ EXP (also open)

PSPACE = {problems solvable in polynomial space}

- $\subseteq$ EXP: only exponentially many states
- $\supseteq$ NP: simulate all executions, take running OR
- open whether either is strict



e.g. Rush Hour is PSPACE-complete [Flake & Baum 2002]
$\Rightarrow \notin P$ if $P \neq NP$ or $NP \neq PSPACE$

Beyond exponential: (not too important)

$EXP(TIME) \subseteq EXPSPACE \subseteq 2EXP(TIME) \subseteq 2EXPSPACE \subseteq \cdots$

double exponential: $2^{2^{n^c}}$

Also $L = LOGSPACE \rightarrow O(\lg n)$ bits of space!

$EXP \subsetneq 2EXP \subsetneq \cdots$ ← time & space hierarchy theorems

$L \subsetneq PSPACE \subsetneq EXPSPACE \subsetneq 2EXPSPACE \subsetneq \cdots$ ↵

Nondeterministic:

- NPSPACE = PSPACE [Savitch 1970] (very useful!) ↗ in general, space bound squares
- NEXP, N2EXP, ...: analogs of NP

What does "as hard as" mean?

Reduction from A to B = poly-time algorithm to convert: instance of A → instance of B (other constraints possible)

such that solution to A = solution to B

⇒ if can solve B then can solve A

$B \in P$, $B \in NP$, ⋮ ⇒ $A \in P$, $A \in NP$, ⋮

⇒ B is **at least as hard as** A (A is a special case of B)

if A→B then A is X-hard ⇒ B is X-hard

- this is a "one-call" reduction [Karp]
- "multi-call" reduction [Turing] also possible: solve A using an oracle that solves B
  - doesn't help much for problems we consider

Examples from algorithms:

- unweighted shortest paths → weighted ($w=1$)
- min-product path → min-sum path ($\lg$)
- longest path → shortest path (negate)
- min-weight k-step path → min-weight path
  (k copies of graph + links between adj. layers)

Almost all hardness proofs are by reduction **from** known hard problem **to** your problem

Examples of hardness proofs:

① Super Mario Bros. is NP-complete

- reduction from 3SAT: can you satisfy (make true) a formula like

$$(x_1 \text{ OR } x_2 \text{ OR } x_3) \text{ AND } (x_5 \text{ OR } (\text{NOT } x_3) \text{ OR } x_4) \text{ AND } \cdots$$

(x_3: variable; $(\text{NOT } x_3)$: literal; x_4: literal; 3 literals per clause)

② Rush Hour is PSPACE-complete

- reduction from NCL (Nondet. Constraint Logic): given directed graph with edge weights $\in \{1, 2\}$, find sequence of edge reversals to reverse a target edge, while at all times maintaining total in-weight ≥ 2 at each vertex
- only need AND vertex & OR vertex

(in fact: OR can be protected: only one input active at once)

- Constraint Logic is a powerful tool for proving hardness of games & puzzles