

TODAY: NP-completeness

- NP-hardness & -completeness

- 3SAT

  - ↳ Super Mario Bros.

  - ↳ 3-Dimensional Matching

    - ↳ Subset Sum

      - ↳ Partition

      - ↳ Rectangle Packing

    - ↳ 4-Partition

      - ↳ Rectangle Packing

      - ↳ Jigsaw puzzles

weak

strong

Recall: (from 6.006)

- $\underline{P} = \{ \text{problems solvable in polynomial time} \}$   
↳ size  $n$       ↳  $n^{O(1)}$

- $\underline{NP} = \{ \text{decision problems solvable in polynomial nondeterministic time} \}$   
↳ output is YES or NO

  - ↳ in  $O(1)$  time can "guess" among polynomial number of choices &

  - if any guess leads to YES.

  - then will make such a guess

"lucky"

- can assume all guessing is done first  
 $\Rightarrow$  equivalent to polynomial-time verifier of polynomial-size certificates for YES answers
- note asymmetry between YES & NO
- problem  $X$  is
  - NP-complete if  $X \in NP$  &  $X$  is NP-hard
  - NP-hard if every problem  $Y \in NP$  reduces to  $X$ 
    - if  $P \neq NP$  then  $X \notin P$  ( $NP - P \rightarrow X$ )
- reduction from problem  $A$  to problem  $B$  = polynomial-time algorithm converting  $A$  inputs into equivalent  $B$  inputs  $A \rightarrow B$ 
  - $\hookrightarrow$  same YES/NO answer
- if  $B \in P$  then  $A \in P$   $\leftarrow A \rightarrow B \rightarrow \text{solve}$
- if  $B \in NP$  then  $A \in NP$
- if  $A$  is NP-hard then  $B$  is NP-hard

How to prove  $X$  is NP-complete:

- ①  $X \in NP$  via nondeterministic algorithm or certificate + verifier
- ② reduce from known NP-complete problem  $Y$  to  $X$

( $\Rightarrow$  any  $Z \in NP \rightarrow Y \rightarrow X \Rightarrow X$  is NP-hard)

- poly-time conversion from  $Y$  inputs to  $X$  inputs
- if  $Y$  answer is YES then  $X$  answer is YES
- if  $X$  answer is YES then  $Y$  answer is YES

3SAT: given Boolean formula of the form:

$$(x_1 \vee x_3 \vee \overline{x_6}) \wedge (\overline{x_2} \vee x_3 \vee \overline{x_7}) \wedge \dots$$

Diagram annotations: Blue arrows point from the word "OR" to the  $\vee$  operators, from "NOT" to the negation bars, and from "AND" to the  $\wedge$  operator. Blue arrows also point from the word "literals" to the individual variables ( $x_1, x_3, \overline{x_6}$ ). A blue bracket under the second clause is labeled "clause".

i.e. formula = AND of clauses  
clause = OR of 3 literals  
literal  $\in \{x_i, \overline{x_i}\}$

└──────────┘ occurrences of variable  $x_i$

is there a variable  $\rightarrow \{T, F\}$  assignment  
such that formula = T (satisfying assignment)

- NP-complete [Cook 1971]
- $\in$  NP: guess  $x_1$  is T or F  
guess  $x_2$  is T or F  
 $\vdots$   
check formula —  $O(\# \text{clauses})$

}  $O(\# \text{variables})$   
nondeterministic

- NP-hard: intuition

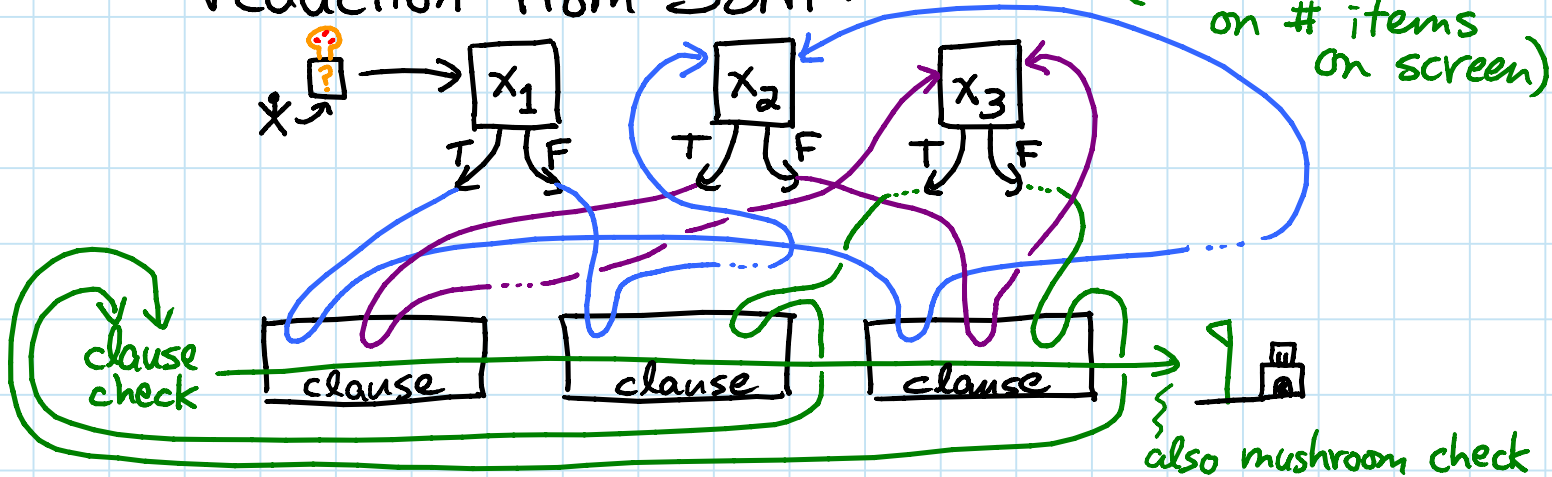
- convert algorithm into a circuit
- convert circuit into a formula
- convert formula into 3CNF (as above)

# Super Mario Bros. is NP-hard

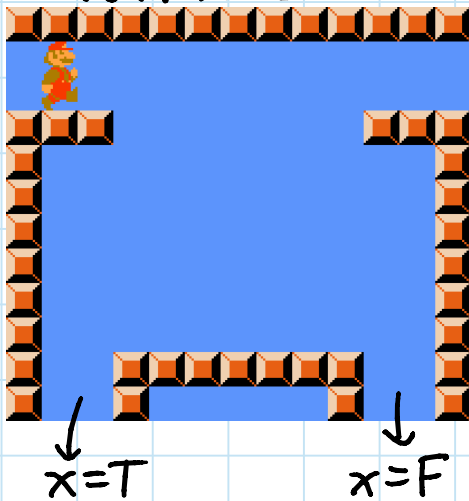
[Aloupis, Demaine, Guo, Viglietta 2014]

- generalized to arbitrary screen size ( $n \times n$ )
- reduction from 3SAT:

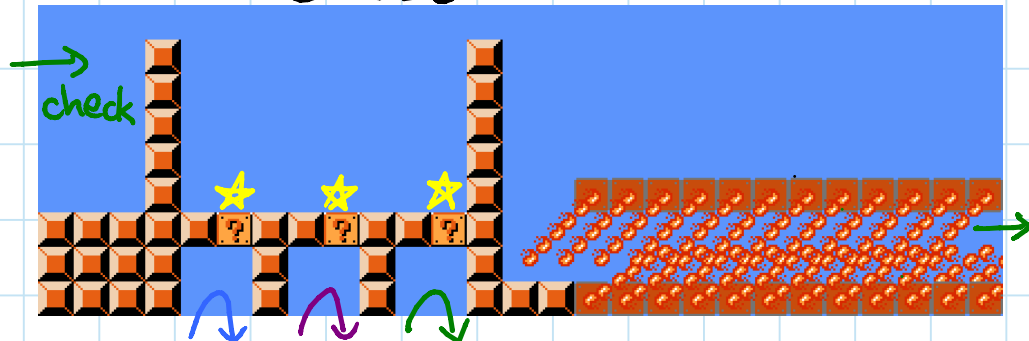
(& remove limits on # items on screen)



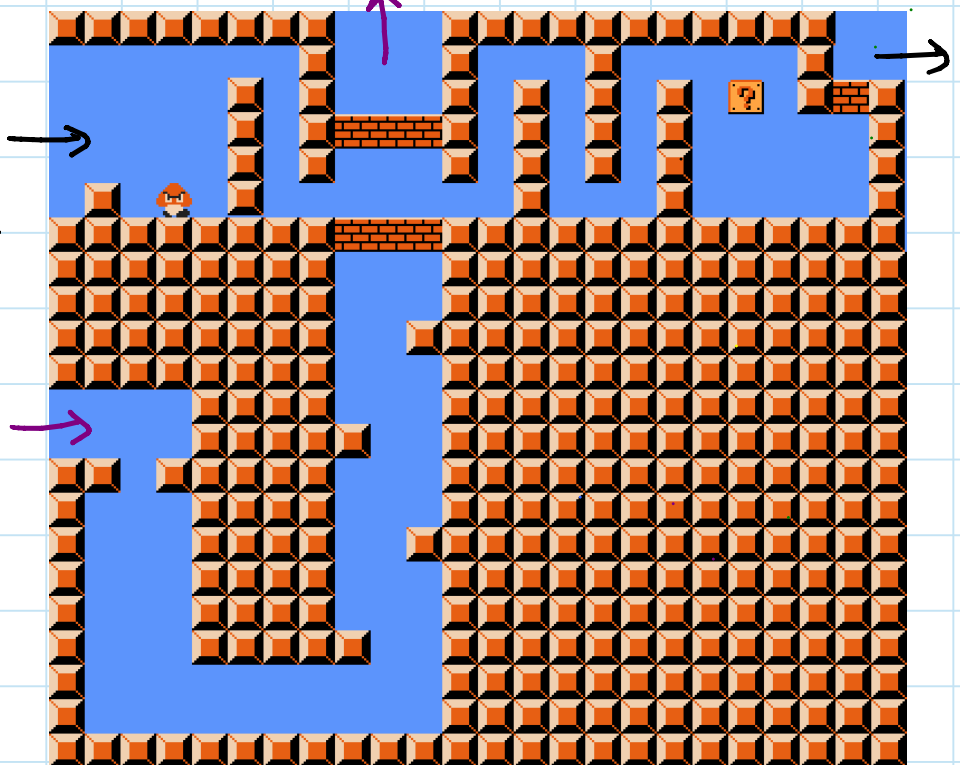
variable:



clause:



cross-over:



For many more cool examples, check out 6.890: "Fun with Hardness"

### 3-Dimensional Matching: (3DM)

given disjoint sets  $X, Y, Z$  each of  $n$  elements, & triples  $T \subseteq X \times Y \times Z$ , is there a subset  $S \subseteq T$  such that each element  $\in X \cup Y \cup Z$  is in exactly one  $s \in S$ ?

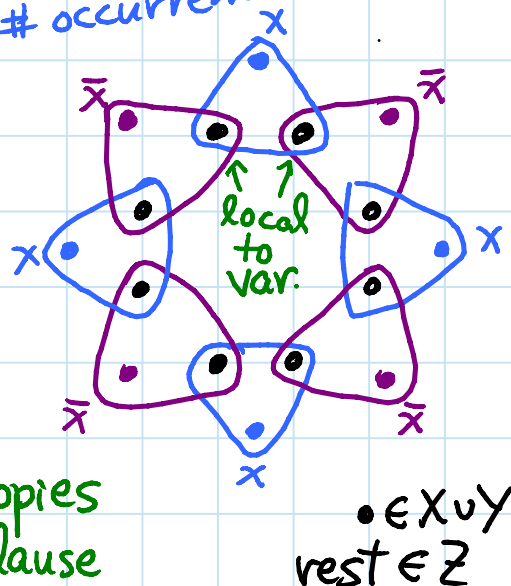
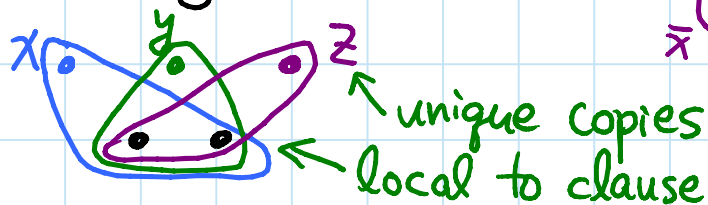
- $\in NP$ : guess which triples  $\in S$  -  $O(T)$  nondet.  
check for exact coverage -  $O(T)$

- NP-hard by reduction from 3SAT:  
[Garey & Johnson 1979 book]  $\rightarrow$  # occurrences of  $x$  or  $\bar{x}$

- variable  $x \rightarrow 2n_x$  chain:

- exactly 2 solutions
- either  $x$ 's or  $\bar{x}$ 's left

- clause  $x \vee y \vee z \rightarrow$



- solvable if  $x$  or  $y$  or  $z$ 's left
- garbage collection:  $\bullet$  all  $x_i$  &  $\bar{x}_i$ 's  
 $\uparrow$  shared (per repeat)

repeated  $\sum_x n_x - \# \text{ clauses}$  times

#  $x$  &  $\bar{x}$ 's left by vars. # covered by clauses

- can cover exactly all unused  $x_i$ 's &  $\bar{x}_i$ 's
- satisfying assignment  $\rightarrow$  3DM  
( $x=T \rightarrow$  leave  $x$ ;  $x=F \rightarrow$  leave  $\bar{x}$ ; satisfy clauses; cover remaining with garbage collector)
- 3DM  $\rightarrow$  satisfying assignment  
( $x$  left  $\rightarrow x=T$ ;  $\bar{x}$  left  $\rightarrow x=F$ ; satisfy clauses)

Subset Sum: given  $n$  integers  $A = \{a_1, a_2, \dots, a_n\}$   
& a target sum  $t$ ,

is there a subset  $S \subseteq A$   
such that  $\sum S = \sum_{a \in S} a = t$ ?

- ENP: guess  $S \subseteq \{1, \dots, n\}$
- pseudopolynomial algorithm via DP (like knapsack)
  - ↳ polynomial in  $n$  & sum of numbers ( $A$ )
- weakly NP-hard by reduction from 3DM
  - ↳ hard when numbers exponential in  $n$  (but still only polynomial number of bits)
- view numbers in base  $b = 1 + \max_i n_{x_i}$ 
  - $\Rightarrow$  never overflow/carry
- triple  $(x_i, x_j, x_k) \rightarrow$  000100100001000
  - $\uparrow \quad \quad \uparrow \quad \quad \uparrow$
  - $i \quad \quad j \quad \quad k$
- $t = 11 \dots 1 = \sum_i b^i$

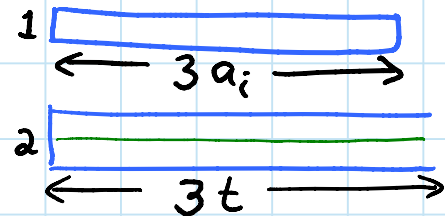
Partition: given  $n$  positive integers  $A = \{a_1, a_2, \dots, a_n\}$ ,  
is there a subset  $S \subseteq A$   
with  $\sum S = \sum (A \setminus S) = \frac{1}{2} \sum A$ ?

- special case of Subset Sum ( $t = \frac{1}{2} \sum A$ )  
 $\Rightarrow \in NP$  & pseudopolynomial algorithm
- weakly NP-hard by reduction from Subset Sum
  - let  $\sigma = \sum_{i=1}^n a_i$
  - add  $a_{n+1} = \sigma + t$  &  $a_{n+2} = 2\sigma - t$   
 $\Rightarrow$  exactly one is  $\in S$  (else  $3\sigma$  vs.  $\sigma$ )
  - $\Rightarrow$  partition must add  $t$  to  $a_{n+2}$   
& add  $\sigma - t$  to  $a_n$

Rectangle packing: given  $n$  rectangles  $R_1, R_2, \dots, R_n$   
& target rectangle  $T$  of area  $\sum_i \text{area}(R_i)$   
can you pack  $R_i$ 's into  $T$  without overlap?

- $\in NP$  because areas match  
 $\Rightarrow$  can only rotate by  $\text{int.} \times 90^\circ$   
 $\Rightarrow$  can guess rotation & integer translation
- weakly NP-hard by reduction from Partition:

- $a_i \rightarrow 1 \times 3a_i$  rectangle  $R_i$
- $t \rightarrow 2 \times 3t$  rectangle  $T$
- $3 > 2 \Rightarrow$  can't rotate  $90^\circ$   
 $\Rightarrow$  packing must find partition





4-Partition: given  $n$  integers  $A = \{a_1, a_2, \dots, a_n\}$ ,  
 is there a partition into  $n/4$  subsets of 4  
 each with the same sum  $t = \sum A / (n/4)$ ?

also works with 3

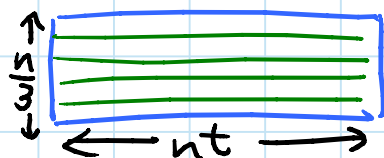
$\in (t/5, t/3)$

- $\in NP$ : guess  $A \rightarrow$  subset mapping
- strongly NP-hard by reduction from 3DM [G&J]
  - $\hookrightarrow$  NP-hard even when number values polynomial in  $n$
- write numbers in base  $r = 100 \cdot \sum (X \cup Y \cup Z)$
- element  $x_i \in X \rightarrow (10, i, 0, 0, 1) = 10r^4 + ir^3 + 1$ 
  - &  $(11, i, 0, 0, 1) \times (n_{x_i} - 1)$  copies
- element  $y_j \in Y \rightarrow (10, 0, j, 0, 2)$ 
  - &  $(11, 0, j, 0, 2) \times (n_{y_j} - 1)$  copies
- element  $z_k \in Z \rightarrow (10, 0, 0, k, 4)$ 
  - &  $(8, 0, 0, k, 4) \times (n_{z_k} - 1)$  copies
- triple  $(x_i, y_j, z_k) \rightarrow (10, -i, -j, -k, 8)$ 
  - $= 10r^4 - ir^3 - jr^2 - kr^3 + 8$
- target sum  $t = (40, 0, 0, 0, 15) = 40r^4 + 15$
- no carries ( $r$  large enough)
- mod  $r \Rightarrow$  use one  $x_i$ , one  $y_j$ , one  $z_k$ , one triple
- $\lfloor \sum / r \rfloor \bmod r \Rightarrow z_k$  & triple match
- $\lfloor \sum / r^2 \rfloor \bmod r \Rightarrow y_j$  & triple match
- $\lfloor \sum / r^3 \rfloor \bmod r \Rightarrow x_i$  & triple match
- $\lfloor \sum / r^4 \rfloor \bmod r \Rightarrow 4 \cdot 10 \rightarrow$  chosen triple  $\in S$ 
  - or  $11 + 11 + 8 + 10 \rightarrow$  unused triple  $\notin S$
- primary (10) form of  $x_i$  (or  $y_j$  or  $z_k$ )  
 must appear in exactly one chosen triple  
 (and elements of triple must all match)



## Rectangle packing:

- strongly NP-hard by reduction from 4-Partition
- $a_i \rightarrow 1 \times n a_i$  rectangle  $R_i$
- $t \rightarrow \frac{n}{3} \times n t$  rectangle  $T$



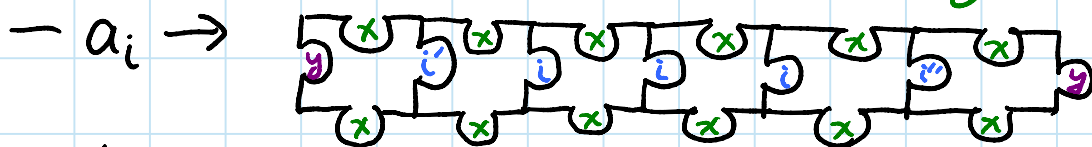
## Jigsaw puzzles:

[Demaine & Demaine 2007]

- model: square tiles (no pattern)  
each side tab, pocket, or boundary  
tabs & pockets must have matching shape  
target rectangular shape



- NP-hard by reduction from 4-Partition:  
(similar to reduction to Rectangle Packing)



- $t \rightarrow$  frame:

