Problem Set 8 - Solutions

1. **Solution:**

   (a) The Monge property cannot be satisfied because shortest paths need not cross.

   (b) This is essentially the bipartite case treated in class. The Monge property holds. Choose, say, a clockwise ordering of the nodes on both cycles.

   (c) Every path in G'' corresponds to a path in G', so shortest paths in G' are at least as short as those in G''. Hence $A'_{i,j} \le A''_{i,j}$. Equality holds if a shortest i-to-j path in G' does not cross the path P.

   (d) Pick an arbitrary node $x \in C_1$ and compute a shortest path tree T in G' rooted at x. Let P_ℓ and P_r be the leftmost and rightmost paths among all paths of T that end at C_2, respectively. It is not hard to see that for every $i \in C_1$ and $j \in C_2$, there exists a shortest i-to-j path in G' that does not cross both P_ℓ and P_r. Let A^ℓ and A^r be the dense distance matrices that correspond to the graphs obtained by cutting G' open along P_ℓ and P_r, respectively. Both A^ℓ and A^r are Monge, and $A'_{i,j} \le {A^\ell}_{i,j}$ and $A'_{i,j} \le {A^r}_{i,j}$. It follows that the column minima of A' are the minimum between the column minima of A^ℓ and A^r which can be found in $O(|C_1| + |C_2|)$ time each using SMAWK.