Problem Set 7

This problem set is due Wednesday, 11/2/2011 at noon.

Problem: Give an $O(n \log n)$–time algorithm to compute an r–division ($O(n/r)$ pieces of size $O(r)$ and boundary $O(\sqrt{r})$) with the additional property that the boundary nodes of each piece lie on a constant number of faces (called "holes"). Note that a face of a piece is not necessarily a face of the graph.

For simplicity, you may assume that the cycle separator theorem achieves perfect balance (meaning that, whenever we apply the separator theorem partitioning V into A, B, S each of the two components $A \cup S, B \cup S$ has weight exactly $w(V)/2$).

Solution: The following algorithm was extracted from a preprint of Christian Wulff-Nilsen (`arXiv:1007.3609v2`), wherein the algorithm is analyzed in great detail.

Regard the entire graph G as a piece with no boundary nodes, split it recursively into two subpieces using a cycle separator, retriangulate each piece, and recurse. The recursion stops when a piece has size at most r.

In the recursion step, we either apply the separator with respect to nodes or with respect to holes. Let $h > 3$ be some constant integer. If at some step the number of holes is larger than h, say $h + 1$ (note that the number of holes increases by one each time we apply the cycle separator theorem), we contract each hole into one single node. Each "hole node" is assigned weight one, all the other nodes are assigned weight zero. Applying the cycle separator theorem to this piece increases the total number of holes by one and it evenly distributes the holes among the two newly created subpieces. Each graph created has at most $(h+1)/2+1 \leq h$ holes. These two separator steps grouped together transform a piece P with h holes into (at most) three pieces each having weight at most $w(V(G))/2$ with at most h holes. We then recurse on each piece.

At this stage, each piece has size at most r and at least $r/4$. The total number of pieces is at most $O(n/r)$. Each piece has at most $O(1)$ holes. The total number of boundary nodes is at most $O(n/\sqrt{r})$ due to the following recurrence (as in G. Frederickson's SICOMP 1987 paper). For each node v let $b(v)$ denote the number of pieces P with $v \in V(P)$ minus one (the number of pieces for which v is a boundary node). Let $B(n,r) = \sum_v b(v)$.

$$
\begin{aligned}
B(n,r) &\leq c\sqrt{n} + 2B(n/2,r) && \text{for } n > r \\
B(n,r) &= 0 && \text{otherwise}
\end{aligned}
$$

Although the total number of boundary nodes is at most $O(n/\sqrt{r})$, some pieces may have more than $c\sqrt{r}$ boundary nodes for a constant c. While there is a piece P with $|\partial P| > c\sqrt{r}$, we apply the cycle separator in P, assigning weight one to each boundary node and weight zero to all the other nodes, again while maintaining a constant number of holes per piece (as described above). The number of new regions due to this procedure is bounded by $O(n/r)$, and the number of new boundary nodes is bounded by $O(n/\sqrt{r})$.