Problem Set 3

This problem set is due Wednesday, October 5 at noon.

1. Let $P := \{p_1, \ldots, p_k\}$ be points in the plane and $\{Q_1, \ldots, Q_t\}$ be a partition of P into t sets. Argue (informally) that there exist disjoint curves in the plane, $C_1, \ldots, C_t$, such that for $i = 1, \ldots, t$, $Q_i \subseteq C_i$.

   Deduce (informally) that there is a suitable function g, such that for every k, the planar grid graph G of size $g(k) \times g(k)$ has the following property: if $u_1, \ldots, u_k \in V(G)$ are sufficiently far apart from each other and from the boundary (i.e. their pairwise distance and distance to the boundary is at least $f(k)$ for a suitable function f) and $\{Q_1, \ldots, Q_t\}$ is a partition of $\{u_1, \ldots, u_k\}$, then there exist disjoint trees $T_1, \ldots, T_t$ in G such that for $i = 1, \ldots, t$, $Q_i \subseteq T_i$.

2. For a vertex v in a graph G and a permutation π_v of its neighbors, define the operation $\texttt{split}(v, \pi_v)$ as replacing v by a path P_v of length degree(v) and connecting each of the neighbors of v to exactly one vertex of P_v in the order specified by π_v.

   Show that for every given integer $k \geq 1$, there exists a planar grid graph augmented by an apex vertex v (i.e. a vertex that is allowed to be connected to every other vertex) such that for any permutation π_v, the graph obtained from applying $\texttt{split}(v, \pi_v)$ contains K_k as a minor.

   Hint: use Problem 1.

Note: Note that applying the `split` operation to every vertex results in a graph of degree at most 3 and note that a grid graph augmented by an apex – a so-called apex-graph – is K_6-minor-free. Hence, there are K_6-minor-free graphs where "no matter how" we split their vertices to reduce to a bounded-degree instance, we will introduce arbitrarily large minors.