Problem Set 10 - Solutions

In this problem set you will develop an algorithm for canceling flow cycles in a given flow assignment. In general graphs this can be done in $O(m \log n)$ time using Sleator's and Tarjan's dynamic trees. You will use the relation between dual shortest paths and circulations to give a linear time algorithm in planar graphs.

1. Let G be a planar graph with non-negative capacities on its arcs. Let ϕ be shortest path distances from f_∞ in G^*. Let θ be the circulation induced by ϕ. That is,

   $$\theta(d) = \phi(\text{head}(d)) - \phi(\text{tail}(d)).$$

   Recall that a residual path is a path whose darts all have strictly positive capacities. Show that there are no counterclockwise residual cycles in the residual graph G_θ.

   Solution: Let T^* be a shortest path tree in G^* rooted at f_∞, and let C be a counterclockwise cycle. Since C encloses at least one face (and by definition does not enclose f_∞), and since T^* is a spanning tree of G^*, there must be a dart d^* of T^* whose dual d belongs to C. By definition of θ, the residual capacity of darts of T^* is zero, so C is not residual.

2. What price function ϕ' would you use to get the same property as in (1), but with no clockwise residual cycles?

   Solution: Reverse the directions of all darts in G^*. Then the argument applies to clockwise cycles in G.

3. Use parts (1) and (2) to give a linear time algorithm that, given a flow assignment γ in G makes γ acyclic by removing all flow cycles in γ. That is, it produces another flow assignment γ' s.t.

   (a) $\gamma' - \gamma$ is a circulation

   (b) for every arc a, $\gamma'((a, 1)) \leq \gamma((a, 1))$

   (c) for any cycle C there is a dart $d \in C$ s.t. $\gamma'(d) \leq 0$

   Solution: Consider the graph G with arc capacities $\gamma(a)$. Let θ_1 be the circulation from part 1. Let γ_1 be the flow obtained from γ by pushing $-\theta_1$ (i.e., $\gamma_1 = \gamma - \theta_1$). γ_1 consists of no counterclockwise cycles, and $\gamma_1((a, 1)) \leq \gamma((a, 1))$ for all arcs a.

   Next consider the graph G with arc capacities γ_1. Let θ_2 be the circulation from part 2. Let γ_2 be the flow obtained from γ_1 by pushing $-\theta_2$ (i.e., $\gamma_1 = \gamma - \theta_1 - \theta_2$). γ_2 consists of no clockwise cycles, and since $\gamma_2((a, 1)) \leq \gamma_1((a, 1))$ for all arcs a, no counterclockwise cycles are introduced. Hence γ_2 is the desired flow.