

In this lecture we study an $O(n \log n)$ -time algorithm for max $s-t$ -flow in planar graph. The algorithm is due to Borradale and Klein with two simplifications and analysis by Erickson.

Settings:

- Non-negative capacities
- t is incident to foo

The algorithm has similarities with MSSP.

The algorithm starts by computing a SP tree T^* in G^* , and pushing the corresponding circulation

$$\hat{f}(d) = \text{dist}_c(\text{foo}, \text{head}(d)) - \text{dist}_c(\text{foo}, \text{tail}(d))$$

in G . Note that for any dart d of T^* , $f(d) = c(d)$, so all darts of T^* are saturated. This property is an invariant of the algorithm.

Consider the primal interdigitating tree T . There is a unique s -to- t path P in T .

We augment the flow along P . Let d be a bottleneck dart of P . d is now saturated so we pivot it into T^* . I.e., we replace the dart of T^* whose head is $\text{head}(d)$ with d . We repeat the process as long as t is reachable from s in T .

MaxFlow (G, c, s, t):

$T^* :=$ SP tree (w.r.t. c) of G^* rooted at f_∞

$T := G \setminus E(T^*)$

$\mathfrak{f} :=$ circulation defined by T^*

$$\mathfrak{f}(d) = \text{dist}_c(f_\infty, \text{head}(d^*)) - \text{dist}_c(f_\infty, \text{tail}(d^*))$$

while s, t are connected in T {

$P :=$ unique s -to- t path in T

$d := \underset{d \in P}{\operatorname{argmin}} \{c(d) - \mathfrak{f}(d)\}$

$\Delta := c(d) - \mathfrak{f}(d)$

foreach $d \in P$:

$$\mathfrak{f}(d) := \mathfrak{f}(d) + \Delta \quad \mathfrak{f}(\text{rev}(d)) := \mathfrak{f}(\text{rev}(d)) - \Delta$$

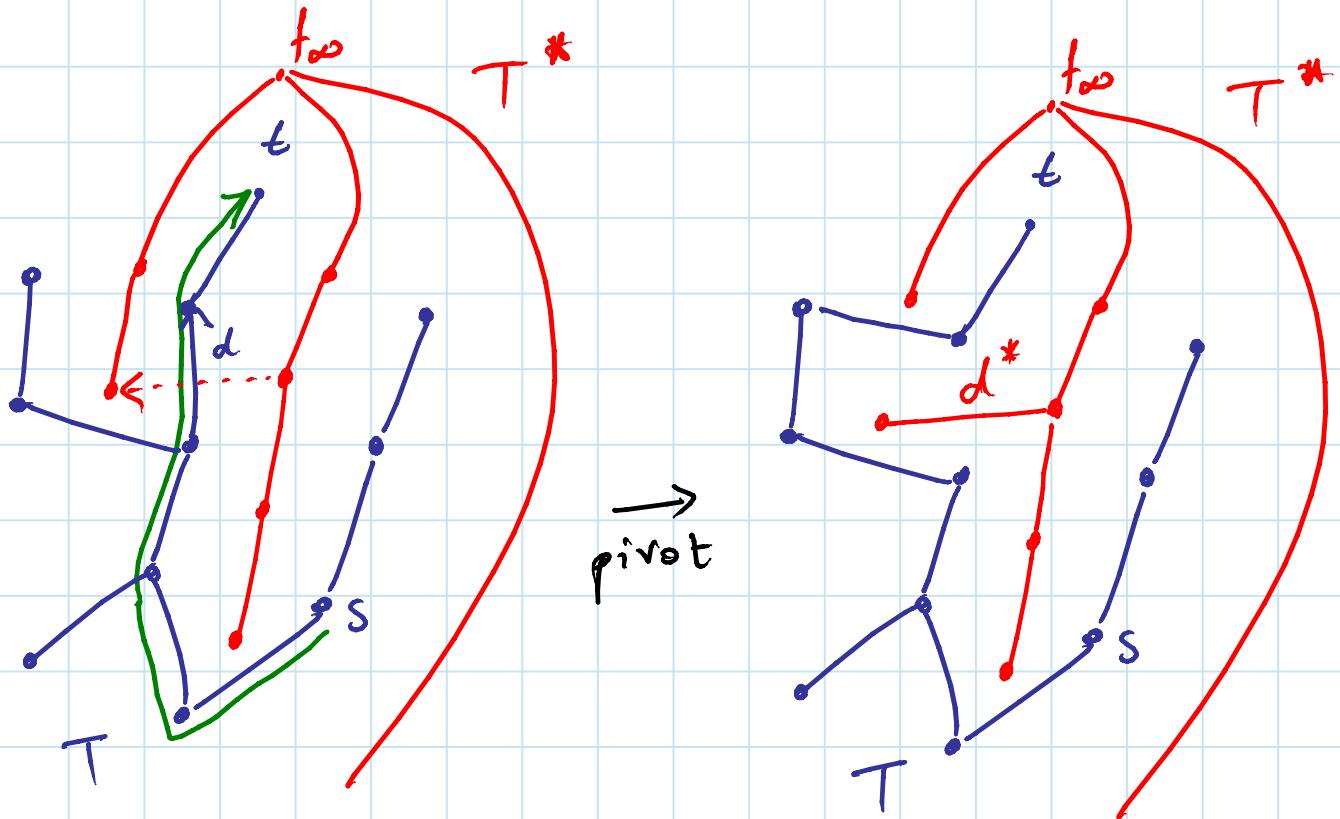
$d' :=$ dart of T^* whose head is $\text{head}(d)$

eject d from T and insert d' into T

eject d' from T^* and insert d into T^*

}

return \mathfrak{f}



flow pushed
along green path
(unique s -to- t path
in T)

dart d
is saturated
and pivots into
 T^*

note: the algorithm terminates if the dart that pivots into T^* happens to be a back edge of T^* . Then there is a saturated dual cycle (primal cut)

it is easy to verify the following invariants:

- (1) \mathcal{T} obeys conservation and respects the capacities c
- (2) An edge is represented in T iff it is not represented in T^*
- (3) The darts of T^* are saturated

The algorithm terminates when s and t are not connected in T . That is, the last pivot made introduced a cycle into T^* . This cycle corresponds to a cut in G , all of whose darts are saturated. Hence, upon termination \mathcal{T} is a maximum flow.

Running time: the original analysis of Borradale and Klein is very complicated. Using dynamic trees, each iteration takes $O(\log n)$ amortized time.

Erickson's analysis: First let us establish the following invariants:

- (4) T^* is an SP-tree w.r.t. residual capacities $C - f$
- Pf: by (3) every dart of T^* has length zero.
by (1) every dart has non-negative length \square

(5) T^* is a SP-tree w.r.t. $C - f'$ for any st-flow f' with the same value as f .

Pf Since f and f' are two st-flows with the same value, they differ by a circulation θ , so $f' = f + \theta$.

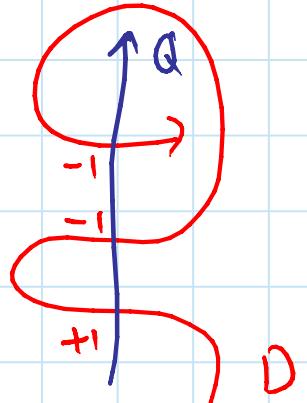
Let φ be the price function that corresponds to θ . By (4) T^* is a SP tree w.r.t. $C - f$. We saw last lecture that the residual capacities after pushing θ are just the reduced lengths w.r.t. φ . Since reduced lengths do not change shortest paths (Lecture 14), T^* is a SP-tree w.r.t. $C - f - \theta = C - f'$ \square

Crossing Numbers:

Let Q be a path and let D be a set of darts. The crossing number of D w.r.t Q is:

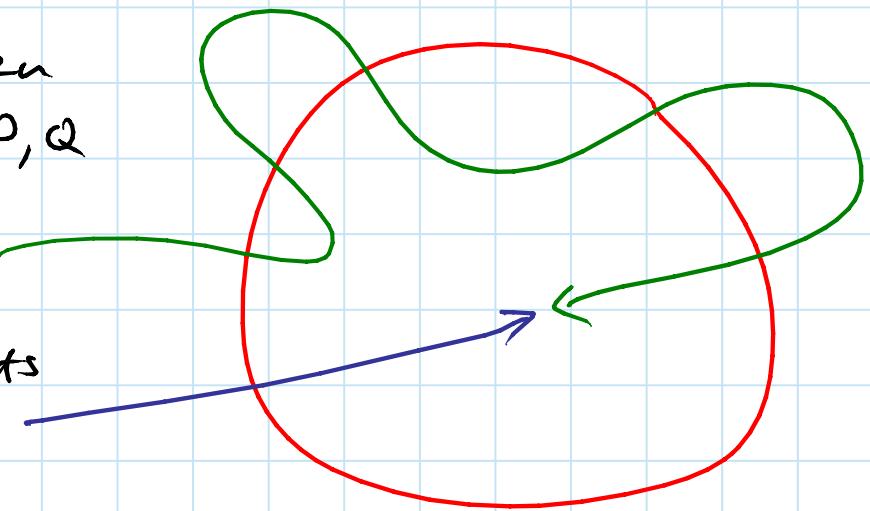
$$\pi_Q(D) = |D \cap Q| - |D \cap \text{rev}(Q)|$$

intuition: think of D as a dual path



Lemma: Let C be a dual cycle. For any two paths P, Q with the same endpoints, $\pi_P(C) = \pi_Q(C)$

Pf: difference between number of times P, Q enter or exit C is $-1, 0$ or 1 , depending on location of endpoints only. \square



Fix an arbitrary s-to-t path Q .

We will use crossing numbers to keep track of the progress made by the algorithm.

Lemma A: Every time $T^*[head(d^*)]$ changes, for some dart d^* , $\pi_Q(T^*[head(d^*)])$ increases by 1.

Pf: consider an iteration. Let d' be the dart that pivots into T^* in that iteration. Let A be the descendants of $head(d')$ in T^* before the pivot is made. Observe that A is the set of heads of darts d s.t. $T^*[head(d)]$ changes in that iteration.

Let R, R' be the paths in T^* to $head(d')$ before and after the pivot, respectively.

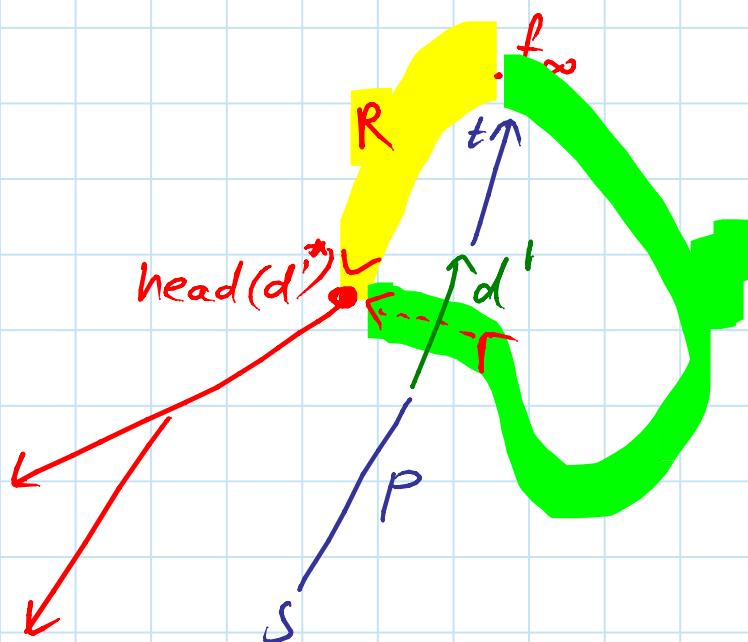
For any dart d s.t. $\text{head}(d) \in A$, R is a prefix of $T^*[\text{head}(d^*)]$ before the pivot. The effect of the pivot of $T^*[\text{head}(d^*)]$ is in replacing the prefix R by R' . Therefore, it suffices to show that $\pi_\alpha(R') = \pi_\alpha(R) + 1$.

Consider C , the fundamental cycle of d' w.r.t. T . C can be written as $R' \circ \text{rev}(R)$.

Since $\pi_Q(C) = \pi_\alpha(R') - \pi_\alpha(R)$, it suffices to show that $\pi_\alpha(C) = 1$.

Since $C \cap P = \{d\}$, $\pi_P(C) = 1$ and hence $\pi_Q(C) = 1$

□



Lemma B: For any face f , $T^*[f]$ is the shortest f_0 -to- f path w.r.t. C among all paths with crossing number $\pi_Q(T^*[f])$.

Pf. Let $\pi_Q(T^*[f]) = i$.

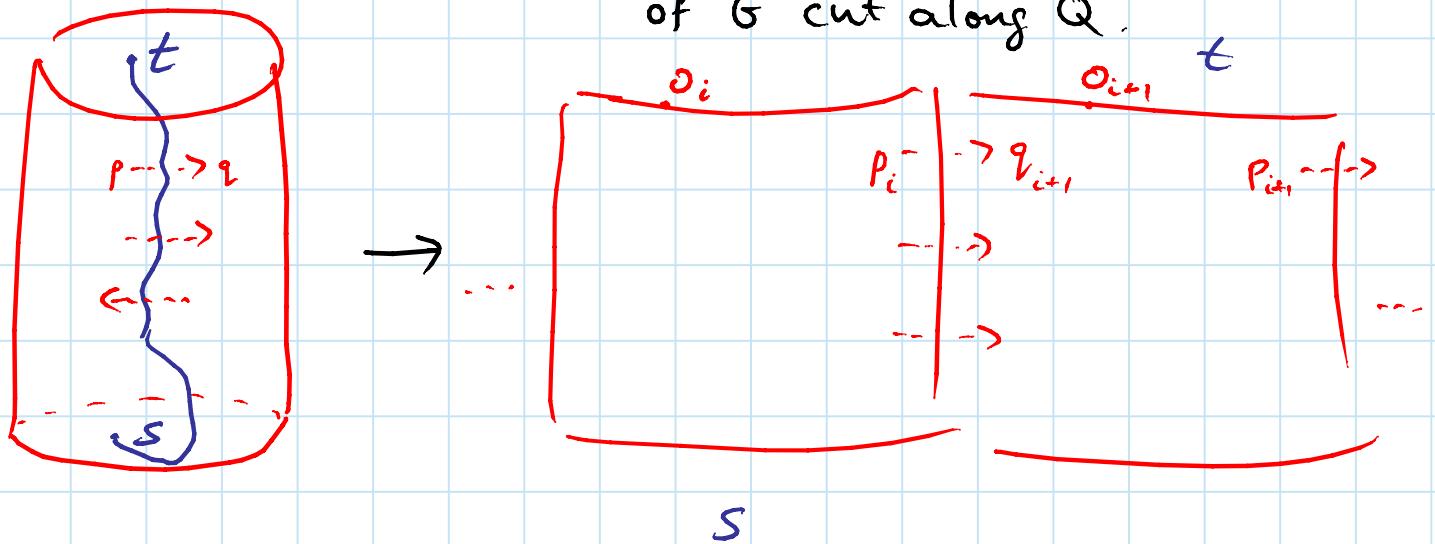
$T^*[f]$ is shortest w.r.t. $C-\delta$, so also w.r.t. a flow T' of value $|f|$ along Q . The length of any path P' with $\pi_Q(P') = i$ w.r.t. $C-\delta'$ is $C(P') - i|\delta'|$, so

$$C(T^*[f]) - i|f| \leq C(P') - i|\delta'|. \text{ Hence,}$$

$$C(T^*[f]) \leq C(P') \quad \square$$

The universal cover $\overline{G^*}$ of G^* is obtained as follows:

infinite strip composed of copies of G^* cut along Q .



delete faces of G^* that correspond to s, t . Cut along Q . Paste an infinite number of copies of the graph thus obtained along the edges dual to Q .

Paths can be lifted and projected between G^* and $\overline{G^*}$ via a natural correspondence between a f -to- g path P' in G^* with $\pi_Q(P') = j$ and f_i -to- g_{i+j} paths in $\overline{G^*}$.

In particular, every shortest path P' in G^* with $\pi_Q(P') = j$ is a projection of some shortest path in $\overline{G^*}$.

for notational convenience, denote f_α by \circ .

We are finally ready to prove a bound on the number of iterations:

Lemma: Each dart pivots out of T^* at most once.

Consider a dual dart $d = fg$.

Assume $d \notin Q$ (the other case is similar).

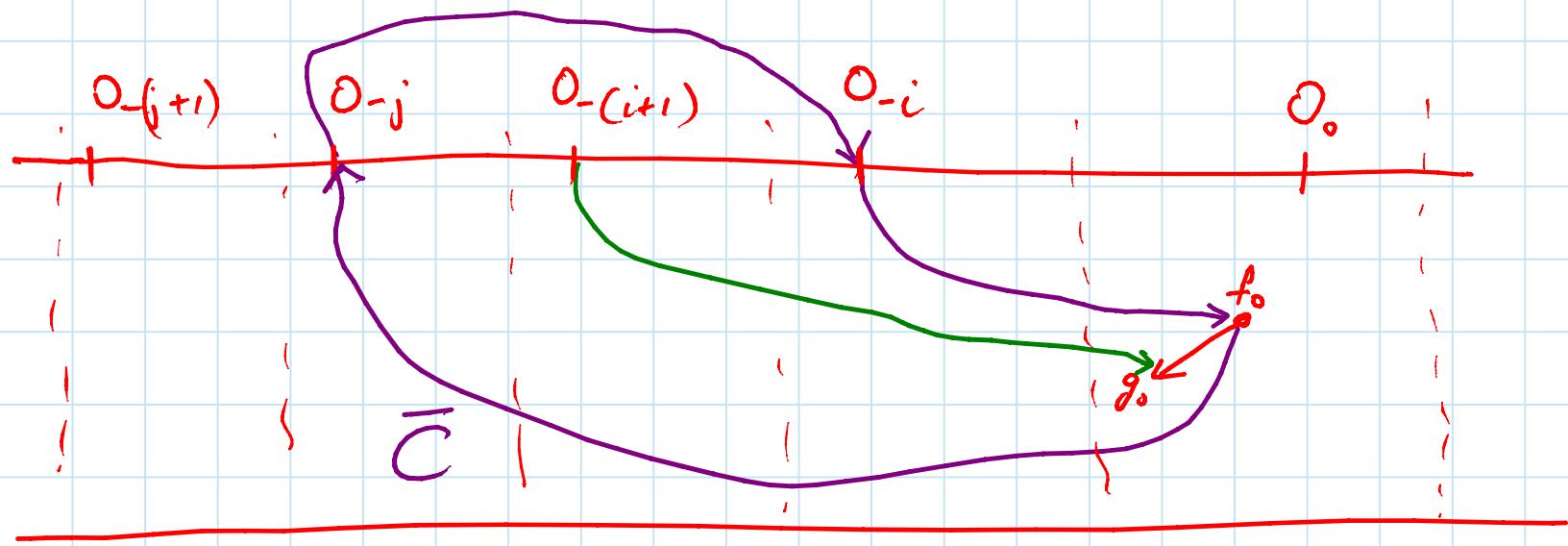
Assume d leaves T^* twice. Let i be $\pi_Q(T^*[\text{head}(d)])$ just before the first time d leaves T^* and let j be $\pi_Q(T^*[\text{head}(d)])$ just before the second time it leaves T^* . By Lemma A, $i < j$. The situation is therefore:

- when $\pi_Q(T^*[\text{head}(d^*)]) = i$, d is the last dart of $T^*[\text{head}(d^*)]$.
- when $\pi_Q(T^*[\text{head}(d^*)]) = i+1$, d is not a dart of $T^*[\text{head}(d^*)]$
- when $\pi_Q(T^*[\text{head}(d^*)]) = j$, d is the last dart of $T^*[\text{head}(d^*)]$.
- when $\pi_Q(T^*[\text{head}(d^*)]) = j+1$, d is not a dart of $T^*[\text{head}(d^*)]$

Consider the cycle \bar{C} composed of the shortest path from O_{-i} to f_0 , the reverse of the shortest path from O_{-j} to f_0 , and a path that connects O_{-i} and O_{-j} outside \bar{G}^* .

By Lemma B, the former two paths correspond to $T^*[\text{head}(d^*)]$ at the time $\pi_Q(T^*[\text{head}(d^*)]) = i$ and j , respectively, so the dart f_0g_0 does not belong to \bar{C} .

Note that $O_{-(i+1)}$ and $O_{-(j+1)}$ are on different sides of \bar{C} . Suppose $O_{-(i+1)}$ is enclosed by \bar{C} . The shortest path from $O_{-(i+1)}$ to g_0 does not contain $f_0 g_0$ (d is not in T^* when $\pi_Q(T^*[\text{head}(d^*)]) = i+1$). Since shortest paths do not cross, g_0 is enclosed by \bar{C} . A symmetric argument for the shortest path from $O_{-(j+1)}$ to g_0 shows that g_0 must be outside \bar{C} , a contradiction. \square



It follows that the algorithm terminates in $O(n)$ iterations. Since each iteration can be done in $\mathcal{O}(\log n)$ amortized time, the total running time is $\mathcal{O}(n \log n)$.

Refs:

- Borradale and Klein - An $O(n \log n)$ algo. for maximum st-flow in a directed planar graph. J. ACM 56(2), 2009.
- Erickson - Maximum flows and parametric shortest paths in planar graphs. SODA 2010
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