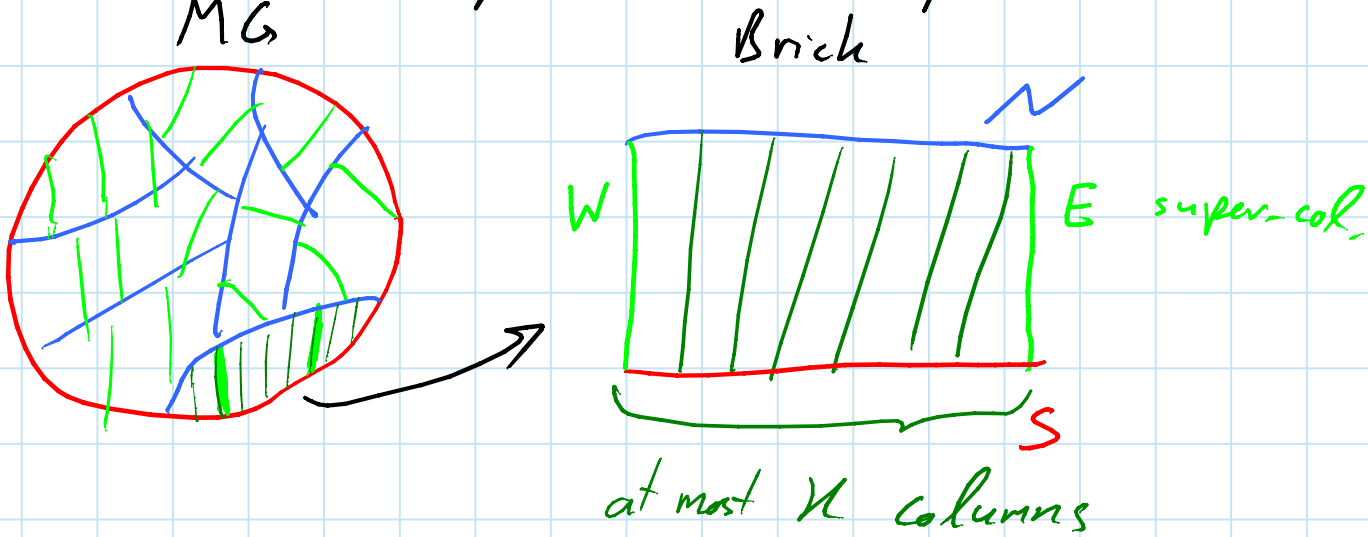
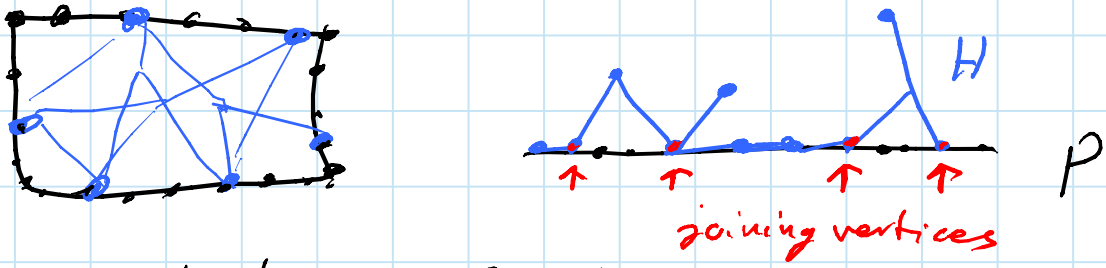
6.889	Lecture 17	November 9, 2011

Recall Mortar Graph-Brick Decomposition



Spanner construction selects a constant number of portals on each brick and adds Steiner trees for every subset of portals.



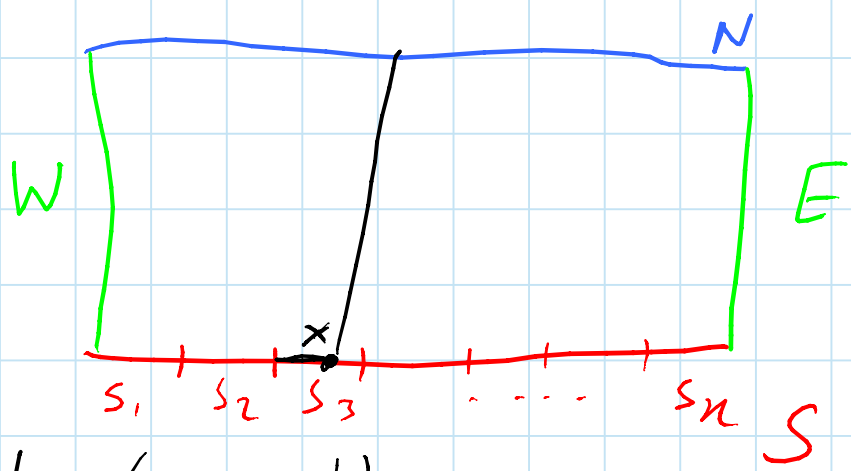
Goal: Show that any forest inside a brick can can be transformed into another forest that spans the same vertices on the boundary but has few joining vertices and approx. the same weight.

Def.: Let $H \subseteq G$ and P be a path in G. A joining vertex of H with P is a vertex of P that is an endpoint of an edge of $H - E(P)$.

Properties of a brick:

- terminals only exist on N or S
- N is a shortest-path
- every proper subpath of S is $(1+\varepsilon)$-short
- $\ell(W)$ and $\ell(E)$ are negligible
  (total weight of all super-columns $< \varepsilon \cdot OPT$)
- $S = S_1 \circ S_2 \circ \dots \circ S_k$ where, for each vertex $x \in S_i[\cdot,\cdot)$:

$$\ell(S_i[\cdot, x]) \lesssim \varepsilon \cdot \mathrm{dist}(x, N)$$

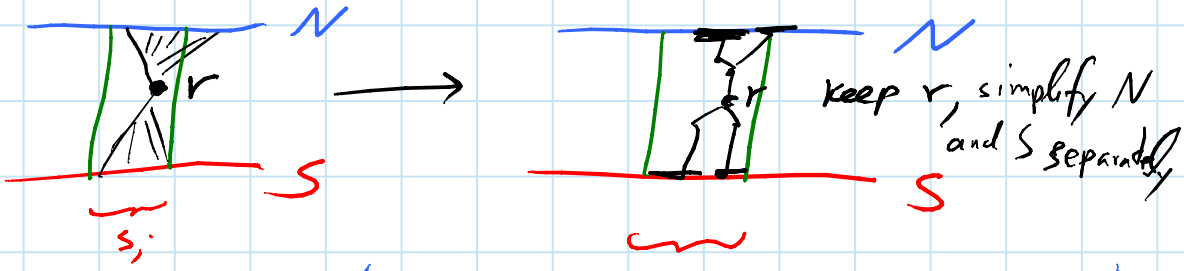


$K = O(\varepsilon^{-3})$

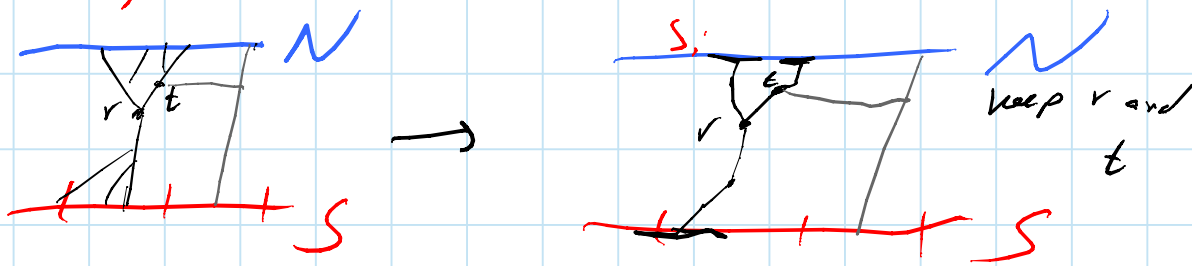
Main Strategy (very rough):

Case 1:

Case 2:



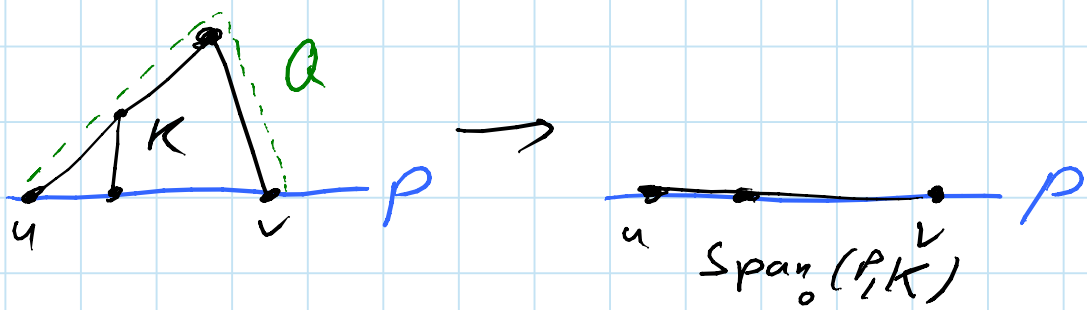
Case 3:



In the following 3 lemmas:

- B a planar embedded graph
- K a subgraph of B
- P an $(1+\varepsilon)$-short path on the boundary ∂B.

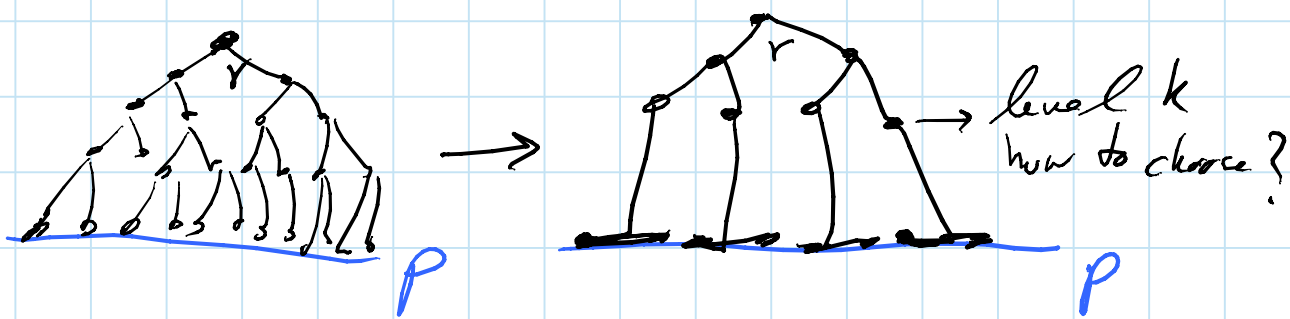
Lemma 1: There is a procedure $\mathrm{Span}_0(P, K)$ that returns a subpath of P spanning $V(K) \cap V(P)$ whose total length is at most $(1+\varepsilon) \cdot \ell(K)$.



Proof: $\ell(K) \geq \ell(Q) \geq \mathrm{dist}_G(u,v) \geq \dfrac{\mathrm{dist}_P(u,v)}{(1+\varepsilon)}$ ✓ ∎

Lemma 2: There is a procedure $\mathrm{Span}_1(P, K, r)$ for $r \in V(K)$ that returns a subgraph of $P \cup K$ such that

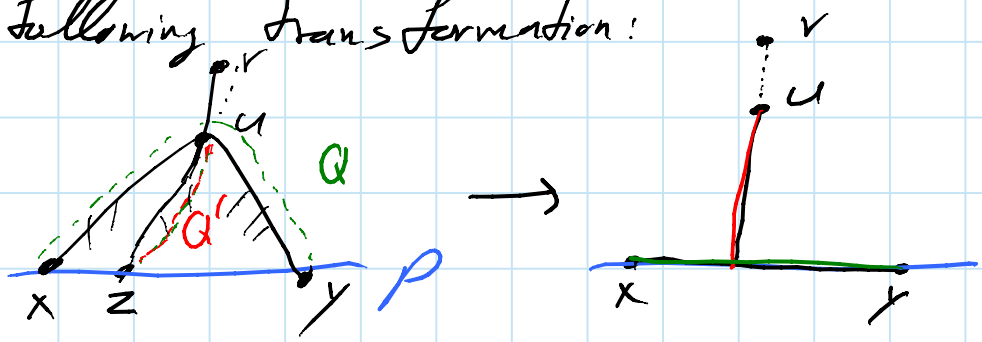
- its length is at most $(1+4\varepsilon)\,\ell(K)$
- it has at most $11\varepsilon^{-1.45}$ joining vertices with P.
- spans $\{r\} \cup (P \cap K)$.



Claim: There is a tree T that is

- rooted at r
- spans all vertices of $K \cap P$
- is binary
- $\ell(T) \leq (1+\varepsilon) \cdot \ell(K)$.

Proof of Claim: Take a subtree of K spanning $K \cap P$ and all leaves on $K \cap P$ and root r. If any vertex has at least 3 children, apply following transformation:



replace subtree of u by the x-y-subpath of P plus the shortest u-to-P path.

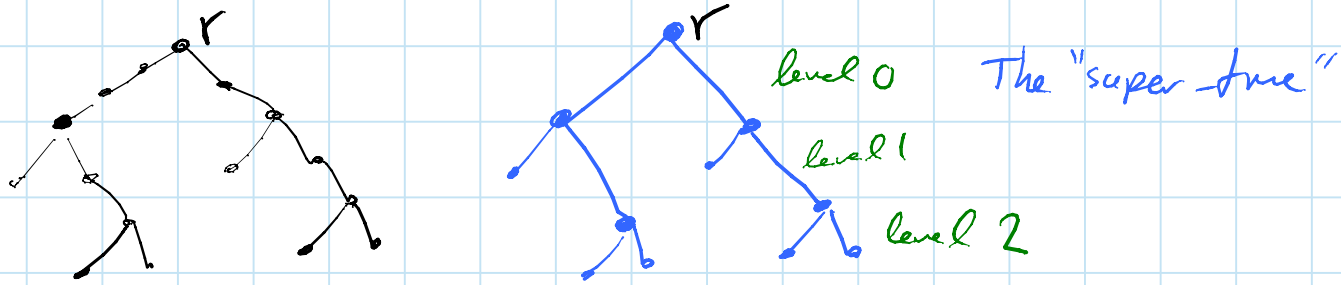
$$\ell(\text{replacement}) = \ell(P_{xy}) + \text{dist}_K(u,P) \leq (1+\varepsilon)\ell(Q) + \ell(Q')$$

$$\leq (1+\varepsilon) \cdot \ell(\text{subtree of } u). \checkmark$$

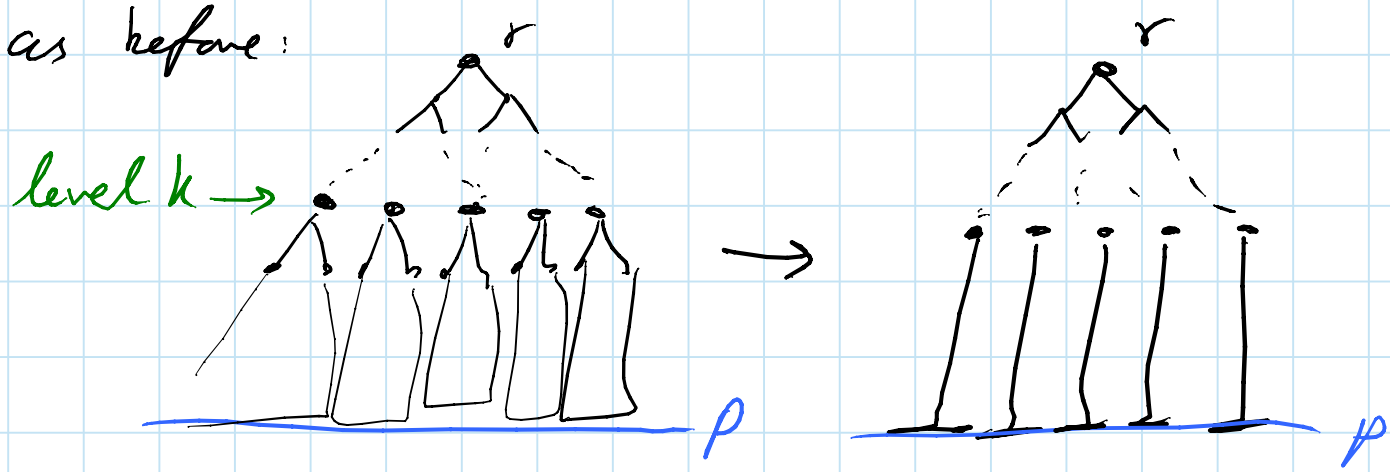
Proof of Lemma 2:

Let a super-edge of T be a maximal subpath of T with internal vertices of degree 2. Endpoints of super-edges are called super-vertices. The level of a super-edge is # of super-edges traversed from the root to its start.

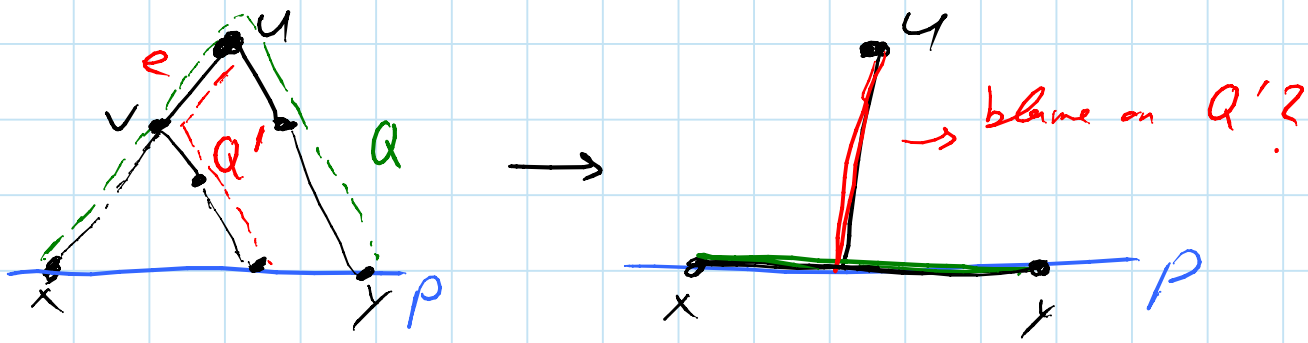
So, we basically ignore vertices of degree 2 (except r).



Select a level k (to be determined) and transform as before:



Note: We can't quite proceed to prove as in previous claim since there is no third path Q' to blame.

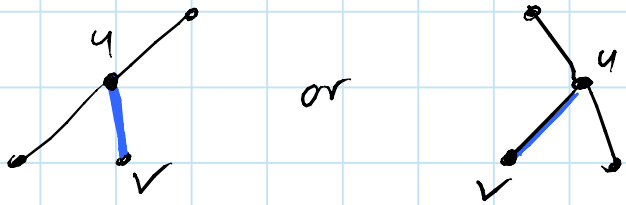


Problem: Super-edge $e = uv$ is blamed twice!

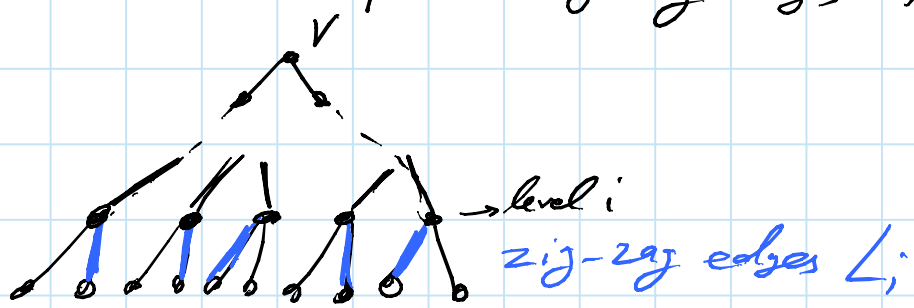
Can we somehow make sure that the total weight of super-edges who are blamed twice is small?

Yes! Choose k appropriately!

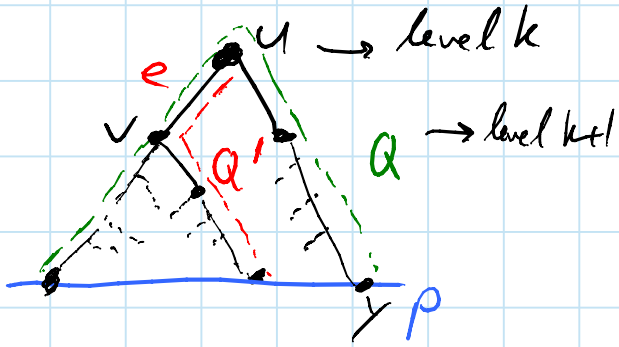
A super-edge uv is called a zig-zag edge if when coming from the parent of u we change direction:



Let L_i denote the set of all zig-zag edges of level i.



Note that we can always designate the bad super-edge e of $Q \cap Q'$ to be a zig-zag edge in L_k. Also, note that $Q \cup Q'$ avoids all zig-zag edges at levels $> k+1$, i.e. avoids $L_{k+2} \cup L_{k+3} \cup \dots$.



Let T' be the tree obtained after transformation:

$$\ell(T') \leq (1+\varepsilon)\left[\ell(T) + L_k - (L_{k+2} + L_{k+3} + \dots)\right]$$

Claim: Let $k_0 = \lceil \log_\phi(\sqrt{5}/\varepsilon + 1) \rceil$ where ϕ is golden ratio. Then there exists $k \leq k_0$ such that $L_k \leq \varepsilon \ell(T) + L_{k+2} + \dots + L_{k_0}$.

Proof of Claim: Otherwise, for every $k = 1, 2, \ldots, k_0$, we have

$$L_k > \varepsilon \ell(T) + L_{k+2} + L_{k+3} + \ldots + L_{k_0}$$

By induction, one can show that $L_1 > \varepsilon \cdot \ell(T) \cdot \mathrm{Fib}(k_0)$ where $\mathrm{Fib}(k_0)$ is the k_0^{th} Fibonacci number, which is greater than $\frac{1}{\varepsilon}$ by choice of k_0. Thus $L_1 > \ell(T)$ ↯

By choosing h according to claim, we have

$$\ell(T') \leq (1+\varepsilon)^2 \cdot \ell(T).$$

Also, the number of super-vertices at level k is at most:

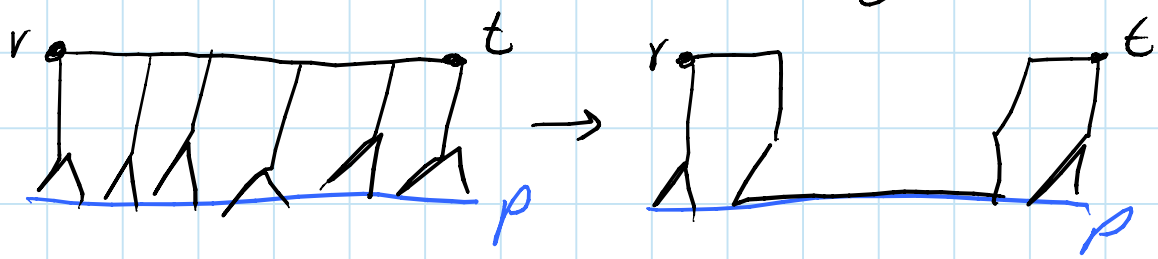
$$2^k \leq 2^{1+\log_\phi(\sqrt{5}(\frac{1}{\varepsilon}+1))} < 11 \cdot \varepsilon^{-1.45}$$

whenever $\varepsilon < 1$. Hence, we obtain desired bound on number of joining vertices.

∎ (of Lemma 2)

Lemma 3: There is a procedure $\mathrm{Span}_2(P, K, r, t)$ that returns a subgraph of $P \cup K$ s.t.

- spans $\{r, t\} \cup (P \cap K)$
- its length is at most $(1 + c_1 \cdot \varepsilon)\ell(K)$
- has at most $c_2 \varepsilon^{-2.5}$ joining vertices with P.



→ see proof in book/paper

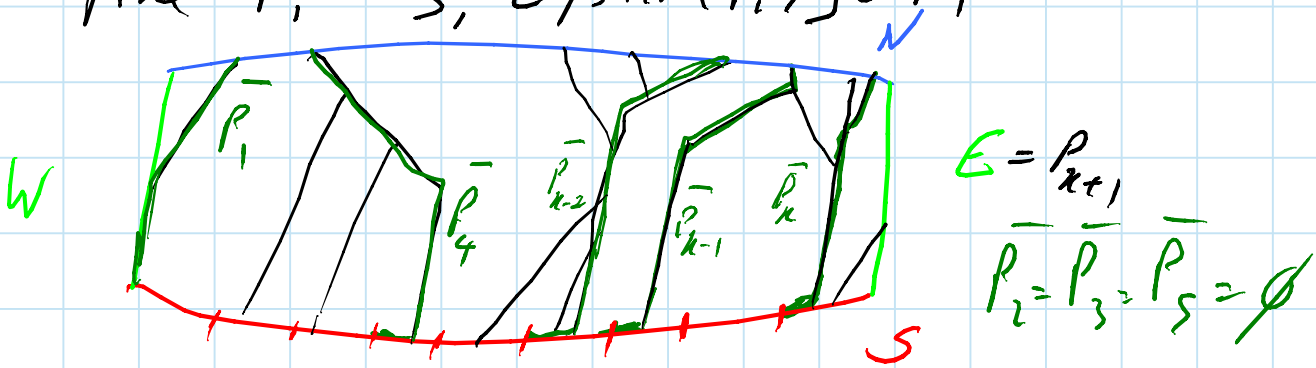
∎

Theorem: Let B be a plane graph with boundary $N \cup E \cup S \cup W$ satisfying the brick properties. Let F be a set of edges of B. There is a forest $\tilde{F}$ of B satisfying the following properties

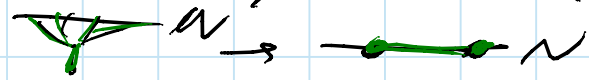
- (F1) If two vertices of the boundary are connected in F then they are connected in $\tilde{F}$.
- (F2) The number of joining vertices of $\tilde{F}$ with N and with S is at most $2C_1 k \varepsilon^{-2.5}$.
- (F2') $\tilde{F}$ includes E and W
- (F3) $\ell(\tilde{F}) \leq (1+c\varepsilon)(\ell(F)+\ell(E)+\ell(W))$.

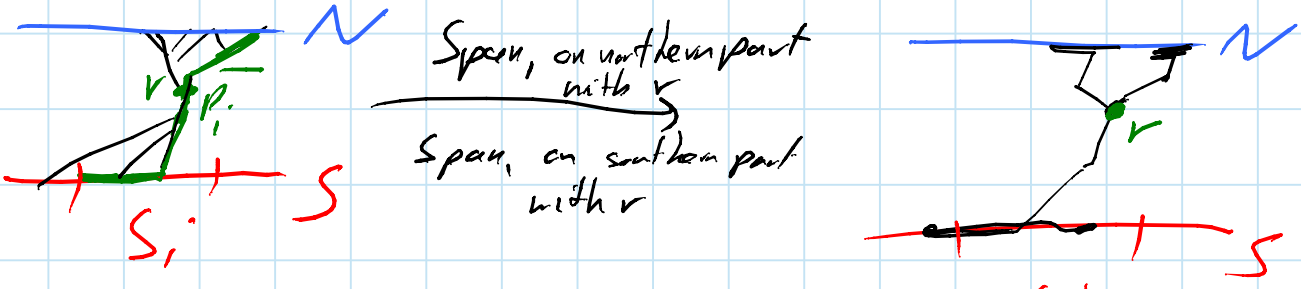
Proof: Define paths $\bar{P}_{k+1}, \bar{P}_k, \ldots, \bar{P}_1$ as follows:

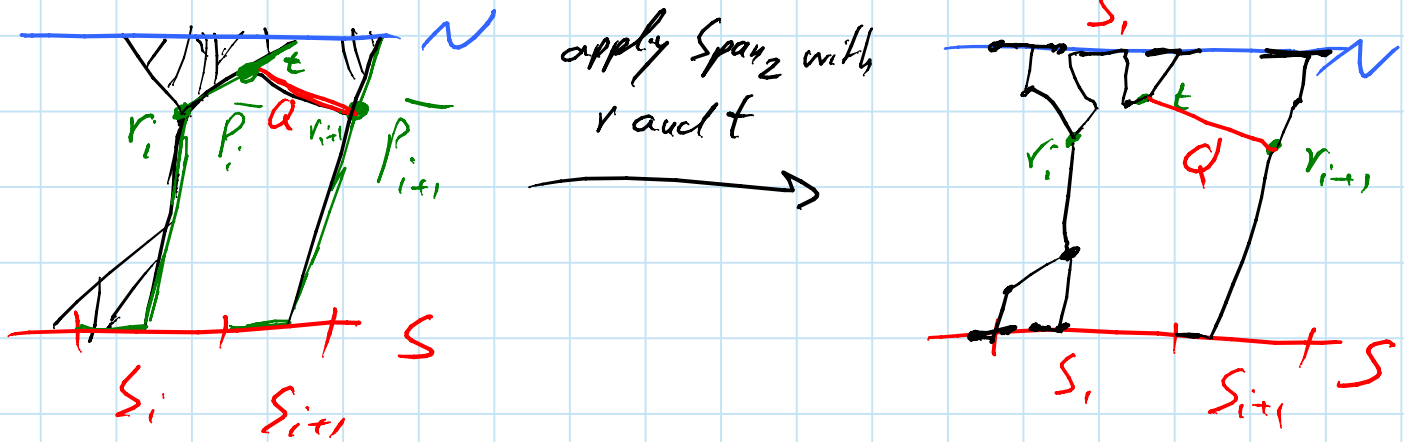
- $\bar{P}_{k+1} := P_{k+1} := E$
- If $F \cup W$ has an S-to-N path that originates in $S_i[\cdot,\cdot)$ and does not intersect $\bar{P}_{i+1}, \ldots, \bar{P}_{k+1}$, let P_i be the rightmost such path.
- Define $\bar{P}_i = S_i[\cdot, \mathrm{start}(P_i)] \circ P_i$



Main Idea:

- get rid of components that have leaves only on N or only on S by applying Span_0 : 

- 

- 

More formally:

- Consider $F \cup W \cup \bigcup_{i=1}^{k} \bar{P}_i$

- Let F' be a minimal forest thereof that contains $\bigcup_{i=1}^{k} \bar{P}_i$ and preserves connectivity among boundary

Note that $\ell(\bar{P}_i) \leq (1+\varepsilon)\ell(P_i)$

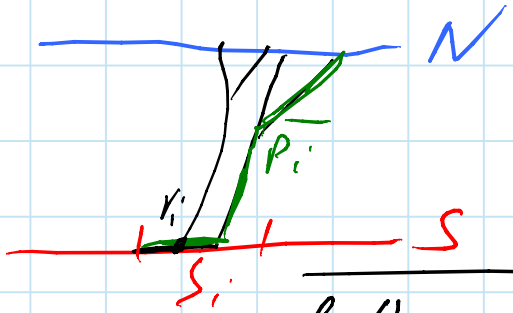
$$\Rightarrow \ell(F') \leq \ell(E) + \ell(W) + (1+\varepsilon)\ell(F)$$

For $i=1,\dots,k+1$ if $\overline{P_i} \neq \emptyset$, let r_i be the first vertex on $\overline{P_i}$ such that there is an r_i-to-N path in F' that avoids other vertices of $\overline{P_i} \cup \dots \cup \overline{P_{k+1}}$.

Claim: For any vertex x of $\overline{P_i}(r_i, \cdot]$, there is no nontrivial x-to-S path that avoids other vertices of $\overline{P_i} \cup \dots \cup \overline{P_{k+1}}$.

Proof: by picture.

Note: r_i could be on S in which case there is a trivial path.



For $i=1,\dots,k$ if there is a path in F' from $\overline{P_i}$ to $\overline{P_{i+1}} \cup \dots \cup \overline{P_{k+1}}$ whose internal vertices are not in $\overline{P_i} \cup \dots \cup \overline{P_{k+1}}$, let Q_i be such a path. Otherwise let $Q_i = \emptyset$.

Claim: Every internal vertex of Q_i has degree 2.

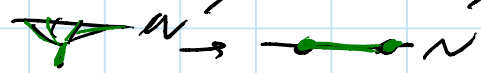
Proof: by picture.

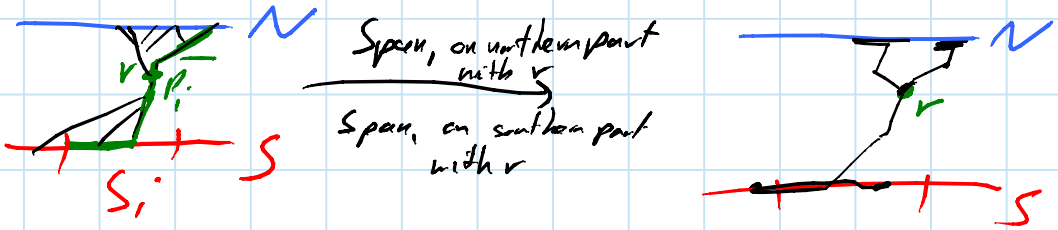
<u>Claim</u>: If there is an integer $j<i$ such that Q_j connects to $\overline{P_i}$ then $\boxed{\text{end}(Q_j) = r_i}$.

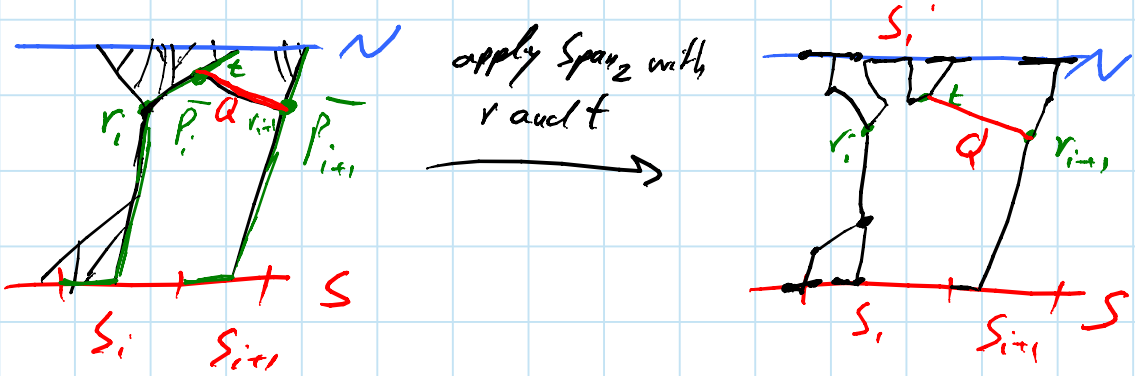
Proof: by picture.

Now we can basically done. Consider each connected component K of $F' - \bigcup_j Q_j$.

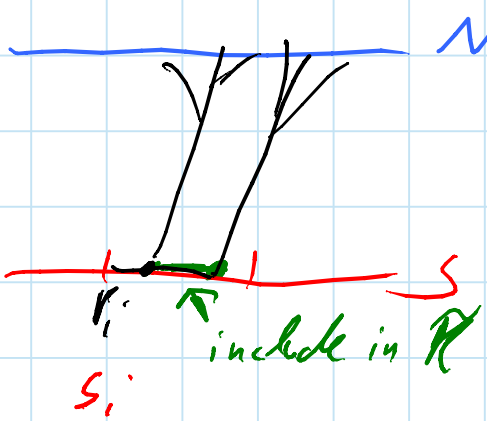
Apply the transformation shown under "Main Idea" to K to obtain $\tilde{K}$ with few joining vertices:

- get rid of components that have leaves only on N or only on S by applying Span_0: 

- 

- 

One more issue: If r_i is on S_i then the northern part still need <u>not</u> span any vertices of S_i. Instead, we simply include the part of S_i that comes after r_i in the solution — it has small weight!



Define $\tilde{F} = \bigcup \tilde{K} \cup \bigcup_j Q_j$.

Need to verify bounds and connectivity requirements (F1), (F2), (F3) of Theorem.

$\longrightarrow$ easy! see book/paper!

References

[BKM09] Glencora Borradaile, Philip N. Klein, and Claire Mathieu. An $O(n \log n)$ approximation scheme for Steiner tree in planar graphs. ACM Transactions on Algorithms, 5(3), 2009.

See also upcoming chapter in Klein's book.