6.889	Lecture 10	Oct. 17, 2011

Bidimensionality 101: [DFHT '05]

Subexponential Parameterized Algorithms in Planar and H-minor-free graphs

	general	H-minor-free	apex-minor-free
vertex cover	1.28^k	$2^{O(\sqrt{k})}$	$2^{O(\sqrt{k})}$
feedback vertex set	3.3^k	"	"
k-path	4^k	"	"
max leaf spanning tree	3.72^k	"	"
connected vertex cover	2^k	(")	"
independent set	W[1]-hard	$2^{O(\sqrt{k})}$ [easy]	"
dominating set	W[2]-hard	" [DFHT '05]	"
connected dominating set	W[2]-hard	" [FLST '12]	"

All blue results via bidimensionality!

Common algorithmic idea:
- check treewidth
- if small → easy ✓
- if large ?
  → use existence of a certificate for large treewidth

Theorem [RS '86, RST '94]: Every graph of treewidth at least $20^{2\ell^5}$ contains an $\ell\times\ell$-grid as a minor. There are graphs of treewidth $\ell^2\log\ell$ that do not contain an $\ell\times\ell$ grid as a minor.

$\Rightarrow$ A minor-closed class of graphs has bounded treewidth if and only if it excludes a planar graph as a minor.

Theorem [RST '94]: Every planar graph of treewidth at least $6\ell-6$ contains an $\ell\times\ell$-grid as a minor.

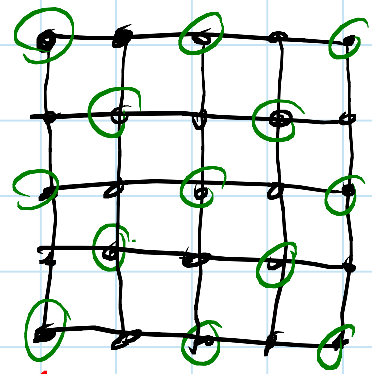
Theorem [DHFT '05]: Every H-minor-free graph of treewidth ℓ contains an $\Omega(\ell)\times\Omega(\ell)$-grid as a minor.

depend on $|H|$

Main Idea of bidimensionality: Use linearity of grid-size in the treewidth.

k-vertex cover in H-minor-free graphs via bidimensionality:

- if treewidth is small, i.e. $\leq c\sqrt{k}$ solve exactly in time $\boxed{2^{O(\sqrt{k})} \cdot n}$.
- otherwise there is a large grid of size $\ell \times \ell$ where $\ell \geq c' \sqrt{k}$, as a minor in G:

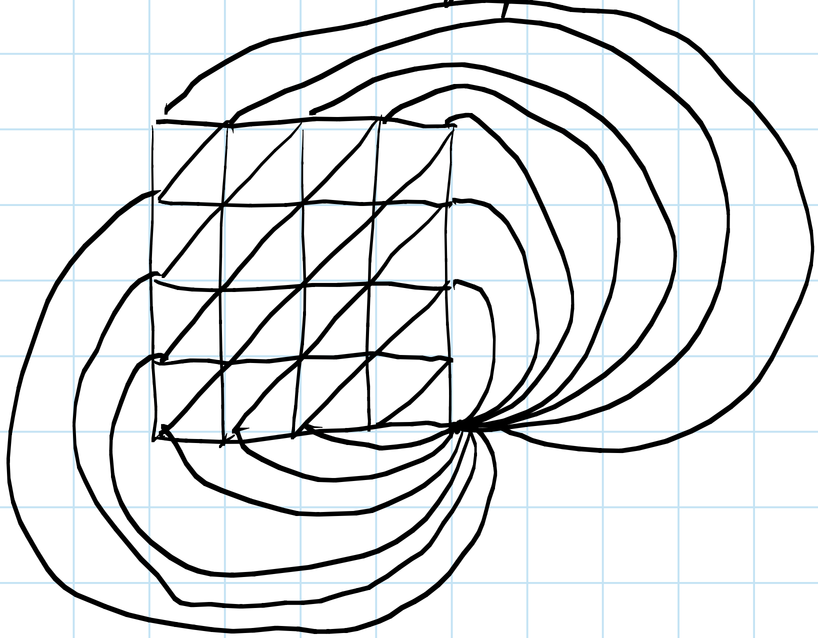


  - but min. vertex cover in $\ell \times \ell$-grid has size $\geq \frac{\ell^2}{2} \geq c'^2 \cdot k > k$ for appropriate c.
  - size of vertex cover can only decrease when taking minors

  $\Rightarrow$ min. vertex cover of G has size $> k$

  $\Rightarrow$ ANSWER IS NO

Theorem [FGT '09]: Every apex-minor-free graph of treewidth $c\ell$ contains the graph T_ℓ as a contraction.



T_5

A problem Π is minor-bidimensional if

- its value does not increase when taking minors.
- its value is at least $\Omega(\ell^2)$ on an $\ell \times \ell$-grid.

A problem Π is contraction-bidimensional if

- its value does not increase when contracting edges
- its value is at least $\Omega(\ell^2)$ on the graph T_ℓ.

Theorem: Every minor-bidimensional (contraction-bidimensional) problem can be solved in time $2^{O(\sqrt{k})} \cdot n$ on all H-minor-free (apex-minor-free) graphs if it can be solved in time $2^{O(t)} \cdot n$ on graphs of treewidth t.

Some minor-bidimensional problems: vertex cover, feedback vertex set, max-leaf, k-path

Some contraction-bidimensional problems: dominating set, r-dominating set, connected dominating set, connected vertex cover, independent set, edge dominating set

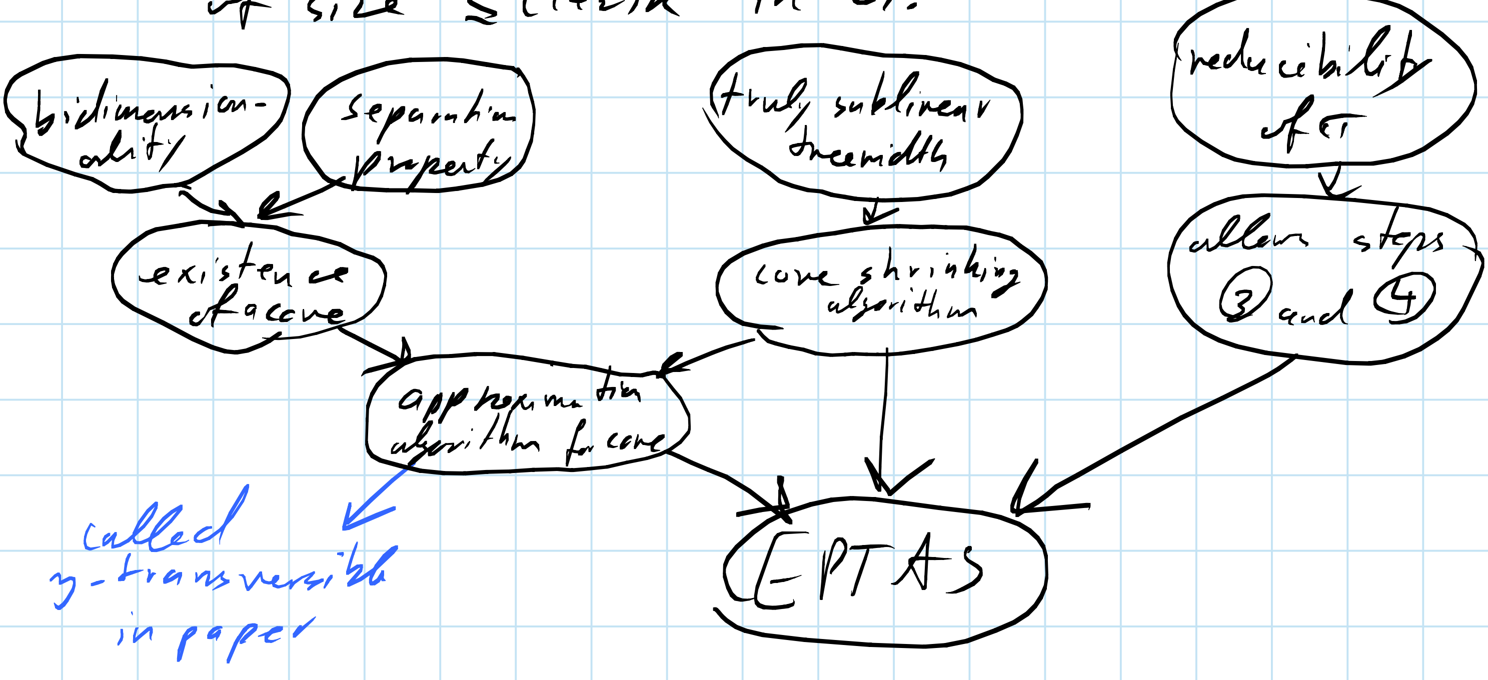
Bidimensionality and EPTAS

[here: FLRS'11, first idea: DH'05]

Let G be an H-minor-free graph, Π a minor-bidimensional problem, OPT an optimal solution to Π in G of value k.
(same works for contraction-bidim. problem on apex-minor-free)

Main idea:

1. find a core $X \subseteq V(G)$ for the problem of size $O(k)$ s.t. $tw(G-X)$ is constant. → depending on Π and H
2. shrink the core to a set $X' \subseteq X$ of size $\leq \varepsilon \cdot k$ s.t. $tw(G-X')$ is $f(\varepsilon)$.
3. solve problem (or a variant thereof) optimally on $G-X'$
4. combine solution of $G-X'$ with X' to obtain solution of size $\leq (1+\varepsilon)k$ in G.



Theorem: Let Π be a reducible minor (contraction-) bidimensional problem with the separation property and H be a (apex) graph. There is an EPTAS for Π on the class of H-minor-free graphs.

→ feedback vertex set, vertex cover, max-leaf spanning tree, connected vertex cover admit EPTAS on H-minor-free graphs (and many more problems)

not minor-bidim. but core can be computed in general.

→ dominating set, r-dominating set, connected dominating set, edge dominating set, independent set, triangle packing, ... admit EPTAS on apex-minor-free graphs.

---

① Existence of a core (Π: minor-bidim., G: H-minor-free)

Separation property: Let (A, B, S) be a separation of G.
Π has separation property if

$$OPT(G[A]) \leq |OPT \cap A| + O(|S|)$$
$$OPT(G[B]) \leq |OPT \cap B| + O(|S|).$$

→ so, separation property allows us to bound size of optimal solution within each part of a separation in terms of the global optimum.

Lemma: Let G be a graph of treewidth t, and $w: V \to \mathbb{R}_0^+$ be a weight function. Then G has a separator S of size at most $t+1$, such that for every connected component C of $G-S$ we have $w(V(C)) \leq \frac{w(V)}{2}$.

$\to$ obtain separation (A, B, S), s.t.

$$\frac{w(V)-w(S)}{3} \leq w(A) \leq 2\,\frac{w(V)-w(S)}{3}.$$

Lemma (existence of a core): [FLST '10]

For minor-bidim. Π with separation property and optimal solution value k on H-minor-free G, we have: There is a subset $S \subseteq V(G)$ and a constant t such that $|S| = O(k)$ and $tw(G-S) \leq t$.

Proof: minor-bidimensionality $\Rightarrow tw(G) \leq d\sqrt{k}$ (since otherwise there would be a large grid and the solution value would be larger than k)

Initialize $S = \emptyset$.

- fix a solution Z of size k and let $w(v) = \begin{cases} 1 & \text{if } v \in Z \\ 0 & \text{o.w.} \end{cases}$
- find a separation (A, B, X) as in previous Lemma.
- set $S = S \cup X$ and recurse on $G[A]$ and $G[B]$. until tw is small.

Let Z_A be opt of $G[A]$ and Z_B opt of $G[B]$.

Note that by separation property $|Z \cap A| = |Z_A| \pm O(|X|)$

Hence $|S| = \mu(G, Z, k) \leq \mu(G[A], Z_A, \frac{k}{3} + d\sqrt{k})$
$+ \mu(G[B], Z_B, \frac{2k}{3} + d\sqrt{k}) + (d\sqrt{k} + 1)$

$\rightsquigarrow$ resolves to $O(k)$

$\blacksquare$

② Truly sublinear treewidth of H-minor-free graphs:

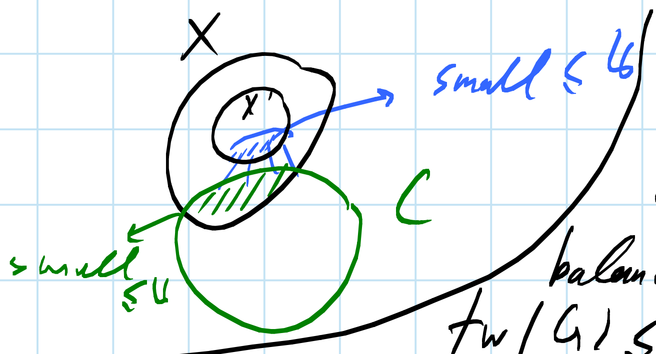
Lemma: Let G be an H-minor-free graph and $X \subseteq V(G)$ such that $tw(G-X) \leq \eta$, where η is a constant. There exists a constant $\beta(H,\eta)$ such that $tw(G) \leq \eta + \beta\sqrt{|X|}$.

Proof. Suppose not. Then G contains a grid-minor of size $(\eta+1)\lceil\sqrt{|X|+1}\,\rceil \times (\eta+1)\lceil\sqrt{|X|+1}\,\rceil \Longrightarrow$

G contains at least $|X|+1$ disjoint $(\eta+1)\times(\eta+1)$ grids as a minor $\Longrightarrow$ one of them is disjoint from X
$\Longrightarrow tw(G-X) \geq \eta+1$ ↯

③ Shrinking a core:

Lemma: Let G be H-minor-free, $X \subseteq V(G)$ with $tw(G-X) \leq \eta$ and $\varepsilon < 1$ be given. There exist $X' \subseteq X$ with $|X'| \leq \varepsilon|X|$ and constant γ such that every connected component C of $G-X'$ has at most γ neighbors in X' and contains at most γ vertices of X.



Proof idea: Clever choice of $\gamma = O(\varepsilon^{-2})$. If $|X| \leq \gamma$, set $X' = \emptyset$. Otherwise define $w(v) = \begin{cases} 1 & \text{if } v \in X \\ 0 & \text{o.w.} \end{cases}$ and find balanced separation (A, B, S). Since $tw(G) \leq \eta + \beta\sqrt{|X|}$, have bound on S. Put $X' = X' \cup S$ and recurse.

Recursion magically resolves to $|X'| \leq \varepsilon|X|$! See paper!

Corollary: Let G be H-minor-free, $X \subseteq V(G)$ with $tw(G-X) \leq \eta$, and $\varepsilon < 1$. There is a constant $J = O(\varepsilon^{-1})$ and an $X' \subseteq X$ s.t. $X' \leq \varepsilon |X|$ and $tw(G - X') \leq J$.

Proof. Apply previous lemma. Note that $\gamma = O(\varepsilon^{-2})$. For any connected component C of $G - X'$, we have $|C \cap X| \leq \gamma$ and so by the truly sublinear treewidth of G:

$$tw(G[C]) \leq \eta + \beta\sqrt{\gamma} = O(\varepsilon^{-1})$$ ∎

---

④ Finding a core:

If Π admits <u>constant-factor approximation</u> → easy:
Use the constant-factor approximation instead of an exact solution as the set Z in core-existence lemma
→ proof of the lemma becomes algorithmic. ✓

<u>Otherwise,</u> use:

Lemma: Let G be H-minor-free and η and integer. There is a constant $c(H)$ and a c-approximation algorithm with running time $n^{O_H(1)}$ for the following problem:
Find a set X of minimum cardinality s.t. $tw(G-X) \leq \eta$.

Proof. Using core shrinking lemma with X and $\varepsilon = \frac{1}{2}$, we know there exists $X' \subseteq X$ and $\gamma(\eta, H)$ with $|X'| \leq |X|/2$ s.t. for every component C of $G - X'$, we have $|C \cap X| \leq \gamma$ and $N(C) \leq \gamma$.

X is <u>smallest</u> set with $tw(G-X) \leq \eta$

$\Rightarrow$ there is a component C of $G-X'$ with $tw > \eta$.

Let $Z = N(C)$. Then $Z \subseteq X'$ and $|Z| \leq k$ and C is connected component of $G-Z$.

Algorithm: Initialize $S = \emptyset$

Try all $\binom{n}{k}$ possibilities for Z to find a connected component C of $G-Z$ with $tw > \eta$.

Note that $tw(G[C]) \leq \eta + O(\sqrt{k})$ (due to truly sublinear tw of G)
$\to$ solve problem optimally on $G[C]$ and let X_C be solution.

Let $S = S \cup X_C \cup \underbrace{N(C)}_{=Z}$.

Repeat on $G - (C \cup N(C))$ as long as its $tw > \eta$.

Let $C_1, \dots, C_t$ be the components found by algorithm.
X must contain at least one vertex in each C_i
$\Rightarrow |X| \geq t$. $\Rightarrow \left|\bigcup_{1 \leq i \leq t} N(C_i)\right| \leq k|X|$

Also for each C_i $|X_C| \leq |X \cap C| \Rightarrow \left|\bigcup_{1 \leq i \leq t} X_{C_i}\right| \leq |X|$

$\Rightarrow |S| \leq (k+1)|X|$

References

[DFHT05] Erik D. Demaine, Fedor V. Fomin, Mohammad Taghi Hajiaghayi, and Dimitrios M. Thilikos. Subexponential parameterized algorithms on bounded-genus graphs and H-minor-free graphs. *J. ACM*, 52(6):866–893, 2005.

[DH05a] Erik D. Demaine and Mohammad Taghi Hajiaghayi. Bidimensionality: new connections between FPT algorithms and PTASs. In *SODA '05: Proceedings of the 16th annual ACM-SIAM Symposium on Discrete Algorithms*, pages 590–601, Philadelphia, PA, USA, 2005.

[DH05b] Erik D. Demaine and MohammadTaghi Hajiaghayi. Graphs excluding a fixed minor have grids as large as treewidth, with combinatorial and algorithmic applications through bidimensionality. In *SODA '05: Proceedings of the 16th annual ACM-SIAM Symposium on Discrete Algorithms*, pages 682–689, Philadelphia, PA, USA, 2005. Society for Industrial and Applied Mathematics.

[FGT09] Fedor V. Fomin, Petr A. Golovach, and Dimitrios M. Thilikos. Contraction bidimensionality: The accurate picture. In *ESA '09: Proceedings of the 17th annual European Symposium on Algorithms*, pages 706–717, 2009.

[FLRS11] Fedor V. Fomin, Daniel Lokshtanov, Venkatesh Raman, and Saket Saurabh. Bidimensionality and EPTAS. In *SODA '11: Proceedings of the 22nd annual ACM-SIAM Symposium on Discrete Algorithms*, pages 748–759, 2011.

[FLST10] Fedor V. Fomin, Daniel Lokshtanov, Saket Saurabh, and Dimitrios M. Thilikos. Bidimensionality and kernels. In *SODA '10: Proceedings of the 21st annual ACM-SIAM Symposium on Discrete Algorithms*, pages 503–510, 2010.

[RST94] Neil Robertson, Paul Seymour, and Robin Thomas. Quickly excluding a planar graph. *J. Comb. Theory Ser. B*, 62(2):323–348, 1994.