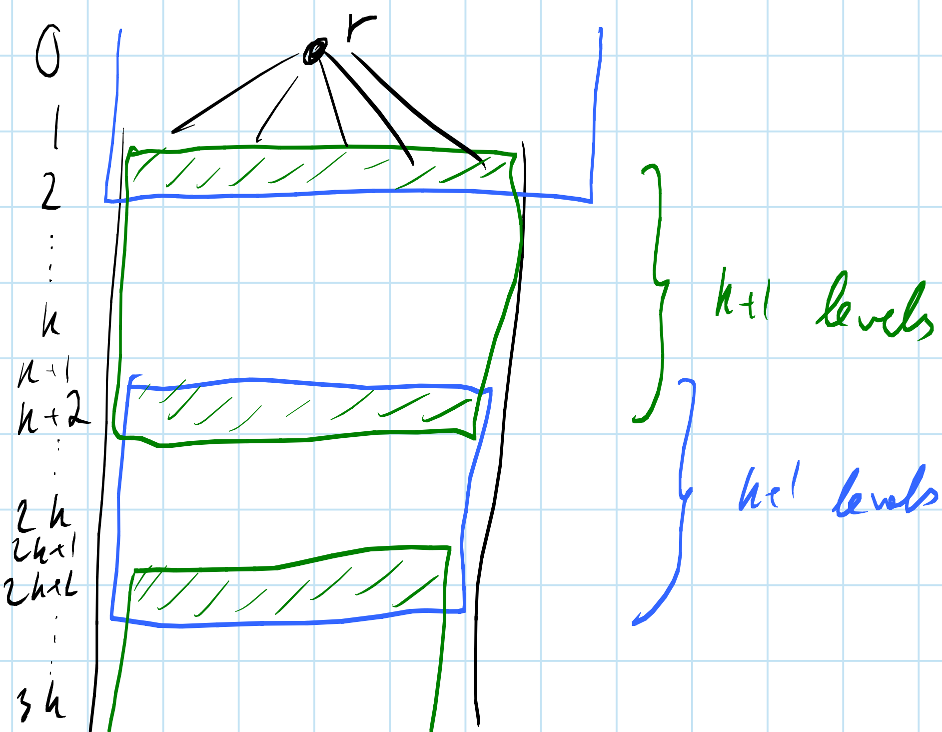
| 6.889 | Lecture 9 | Oct. 12, 2011 |
|---|---|---|

Recall essence of Baker's technique:

- perform BFS from some vertex $r$
- for some parameter $k$ (say $k = 1/\varepsilon$), consider slices of $k+1$ consecutive levels of BFS-tree
- each slice has treewidth $O(k)$
- $k$ possible shifts of slices, namely, for each $i \bmod k$.
- the boundary of a shift are exactly the vertices at levels $i \bmod k$.

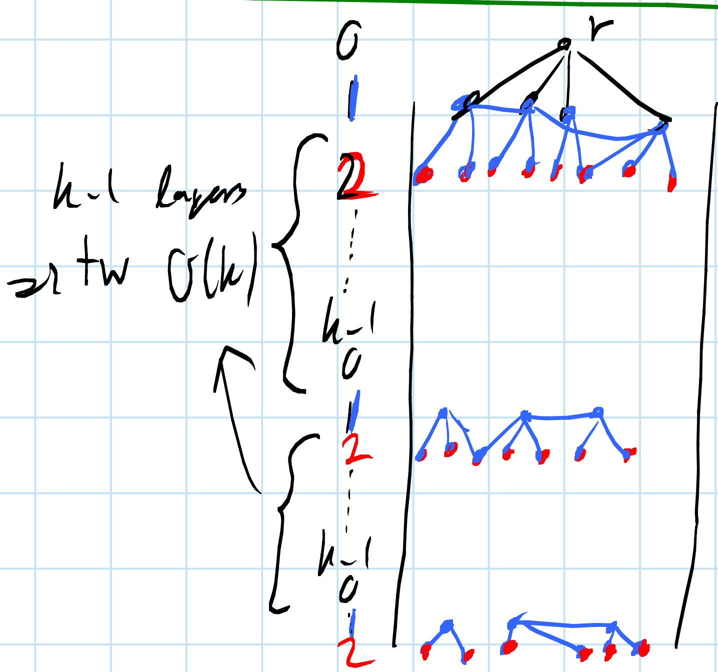
$\Rightarrow$ one of the boundaries contains small ($1/k$) portion of an optimal solution.

boundary are vertices at levels 2 mod $k$.



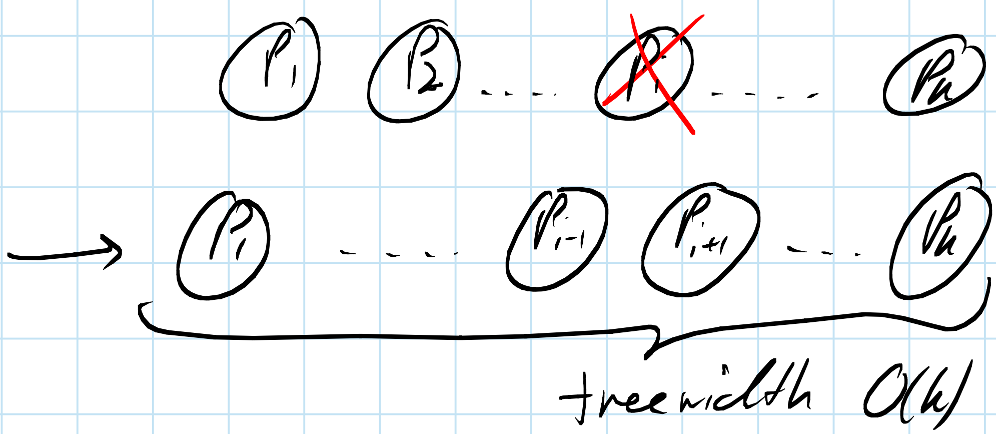
Same idea, different perspective:

$\boxed{\text{Label each vertex according to its layer mod } k \qquad \mathrm{label}(v) := \mathrm{dist}(r,v) \bmod k}$




Also, label each edge with the label of its endpoint closest to r.

=> We obtain a partition of the vertices or edges of G into k sets such that deleting any one part results in a graph of treewidth O(k).




We obtain a simplifying decomposition.

Consequence: can easily obtain PTAS for hereditary maximization problems like independent set, max-cut, maximum P-matching, ...

- for $k = 1/\varepsilon$ compute partition $P_1, \dots, P_k$ as above
- for some $q$, $|OPT \cap P_q| \leq \frac{1}{k} \cdot |OPT| \leq \varepsilon \cdot |OPT|$
- hence, $|OPT \cap (G - P_q)| \geq (1 - \frac{1}{\varepsilon}) |OPT|$
- return $\max_i \{ |MIS(G - P_i)| \}$

easy because of bounded tw

---

Similar for some minimization problems, e.g. vertex cover.
However, this approach is simpler but weaker than Baker's original technique!

Baker's technique applies to other problems such as dominating set or r-dominating set as well.
→ have slices overlap on some r levels.
→ is not captured by deletion decomposition view!

---

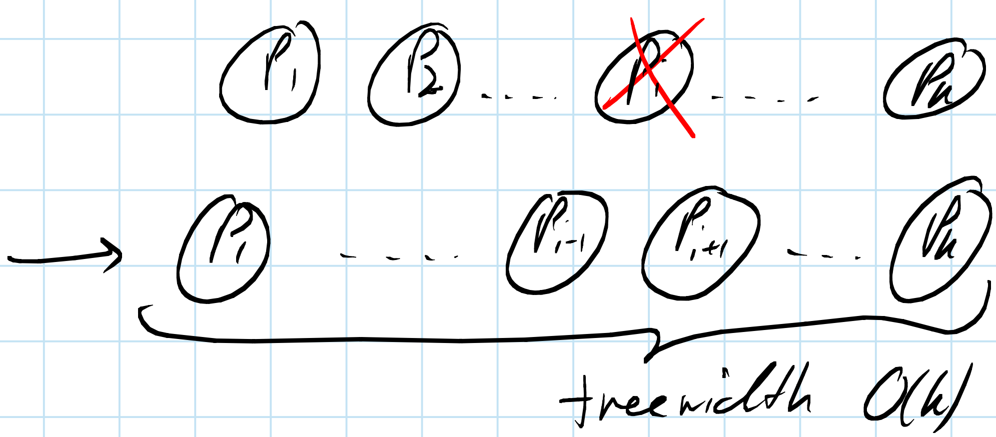
Also: By putting $k=2$, we easily obtain 2-approximations for a lot of problems, e.g. coloring.

**Theorem (DHK '05):**

For a fixed graph $H$, there is a constant $c_H$ such that, for any integer $k \geq 1$ and for every $H$-minor-free graph $G$, the vertices of $G$ (or the edges of $G$) can be partitioned into $k+1$ sets such that any $k$ of the sets induce a graph of treewidth at most $c_H \cdot k$. Furthermore such a partition can be found in polynomial time.

**Corollary:** There is a 2-approximation for Coloring $H$-minor-free graphs

**Corollary:** There is a PTAS for independent set, vertex cover, max-cut, max $P$-matching in $H$-minor-free graphs.




**Note:** There is also a PTAS for $(r-)$dominating set in $H$-minor-free graphs [Grohe '03] but uses direct generalization of Baker's technique.

Why does Baker's idea work?

$\boxed{\text{Planar graphs of small diameter have small treewidth.}}$

Motivates the definition of local treewidth:
We want the treewidth of small neighborhoods be small.

Let $N_r(v)$ denote the $r$-neighborhood of a vertex $v$.

Define the local treewidth of a graph $G$ by


$$ltw_G(r) := \max_{v \in V(G)} \{ tw(G[N_r(v)]) \}$$

A class of graphs $\mathcal{C}$ has bounded local treewidth if for all $G \in \mathcal{C}$ we have

$$ltw_G(r) \leq f(r)$$

where $f$ is a function depending only on the class $\mathcal{C}$.
If $f(r) = \lambda \cdot r$ for some constant $\lambda$, then the class has linear local treewidth.

---

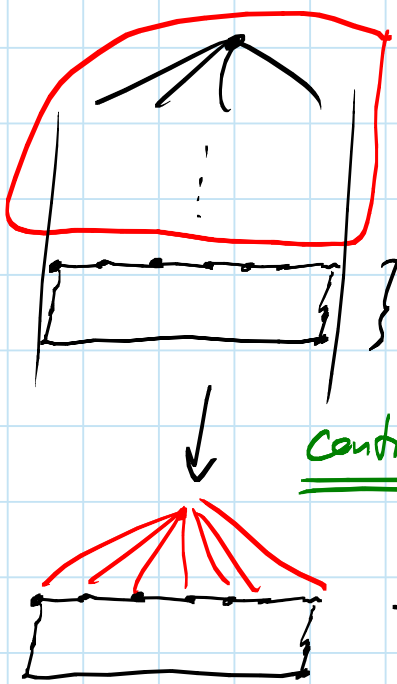
Saw in last lecture that planar graphs have linear local tw.

Bounded degree graphs of degree $d$ have bounded local tw with $f(r) = d^r$.

Consider any class of graphs of bounded local tw.
Does Baker's approach work?

No! There is no PTAS for independent set on bounded degree graphs unless P = NP.

Consider k consecutive layers of a BFS tree.



} k layers but large diameter!

↓ contract all previous layers (delete whatever is after)

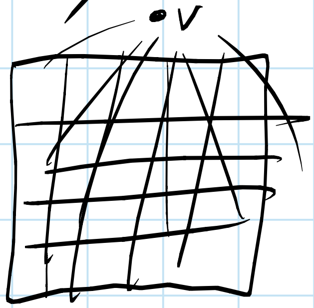
→ diameter 2k+1 ⇒ small tw!

So, it does work if our class is minor-closed and has bounded local treewidth.

Note: Upon contracting a connected subgraph in a bounded degree graph, the graph is no longer of bounded degree and not necessarily of bounded local treewidth anymore.

So, local treewidth is not preserved under minors.

Consider a $k \times k$ grid with an apex $v$ connected to every vertex:




$N_1(v) = G$, $tw(N_1(v)) = k+1$ $\Rightarrow$ !

$\Rightarrow$ a minor-closed class of graphs of bounded local treewidth cannot contain all apex graphs.

Theorem [Eppstein '00]:
A minor-closed class of graphs has bounded local treewidth if and only if it is apex-minor-free. $\rightarrow$ see exercises!

[Demaine, Hajiaghayi '04]: In this case the class has in fact linear local treewidth.
$\hookrightarrow$ complicated proof using RS-decomp.

$\Rightarrow$ Deletion decomposition theorem and Baker's technique work correctly on all apex-minor-free classes of graphs.

Theorem [Robertson-Seymour decomposition]:

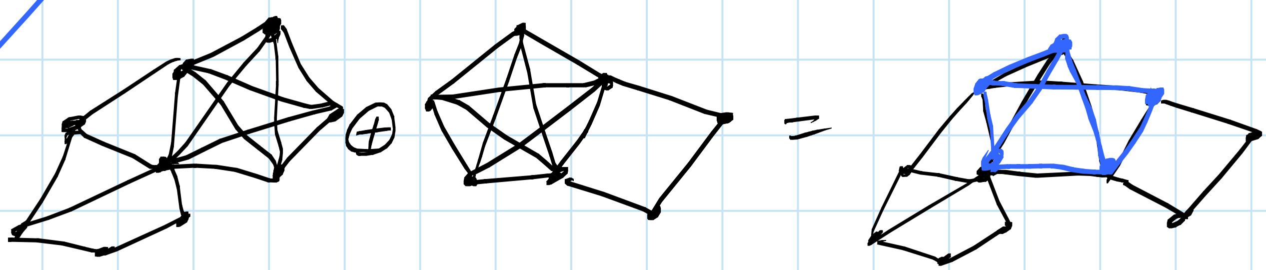
Every $H$-minor-free graph $G$ can be written as an $h$-clique-sum of $h$-almost embeddable graphs $G_1, \dots, G_\ell$, where $h$ is a constant depending only on $H$:

$$G = G_1 \oplus G_2 \oplus \dots \oplus G_\ell$$

Furthermore, for each $G_i$, the vertices of the clique that connect it to $G_1 \oplus \dots \oplus G_{i-1}$ are completely contained in the apex set of $G_i$. → can be found in $O_H(n^3)$ [Kawarabayashi, Wollan '11]

h-clique-sum of $G_1$ and $G_2$:

- pick a clique $C_1$ in $G_1$ of size $t \leq h$
- " " $C_2$ in $G_2$ " " "
- identify (JOIN) $C_1$ and $C_2$ to obtain $G_1 \oplus G_2$
- delete some edges of the clique

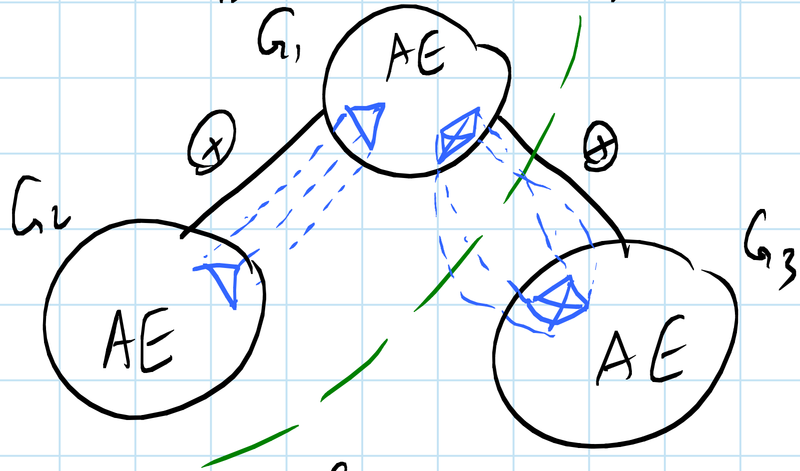


This operation is not well-defined → can have several outcomes.

Note: $tw(G_1 \oplus G_2) \leq \max\{tw(G_1), tw(G_2)\}$

see next page

Indeed, can think of $G$ as having a tree decomposition where (the closure of) every bag is almost embeddable and the intersections of bags are the clique-sums.

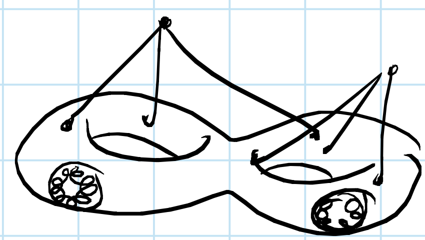



Note: The edges we remove from a clique-sum are called virtual edges. They do not exist in $G$ but the point is that even if they existed, the parts $G_1, \dots, G_\ell$ would be almost embeddable. Indeed, the fact that these are cliques is key in many proofs.

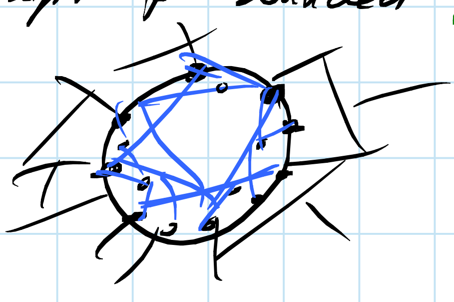
---

h-almost-embeddable:

- bounded-genus graph of genus $\le h$
+ at most $h$ apices
+ at most $h$ vortices



Vortex: a face of the bounded-genus graph in which we have a graph of bounded pathwidth.



Theorem [Grohe '03]: The class of all minors of an apex-free h-almost-embeddable graph has linear local treewidth.

↳ has to deal with vertices

↳ has to deal with minors of AE-graphs

## Proof of deletion decomposition theorem:

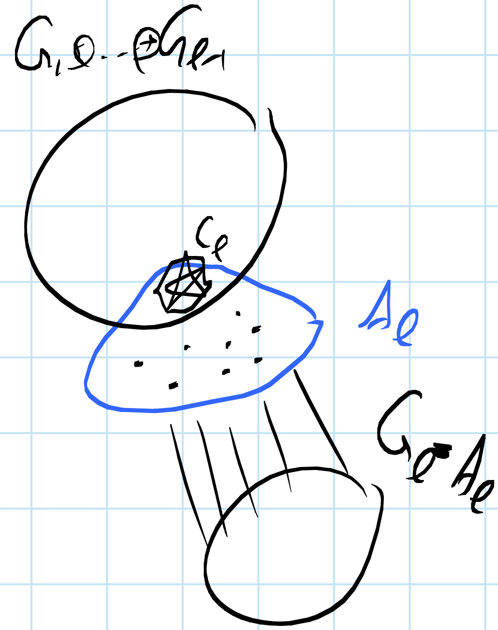
Let $G = G_1 \oplus G_2 \oplus \dots \oplus G_\ell$. Proof by induction on $\ell$.

If $\ell = 1$, use Grohe's theorem on the apex-free part and assign arbitrary labels to the apices. Since the number of apices is bounded by $h$, they can be added to all the bags of a tree decomposition while increasing the width only by a constant.

For $\ell \geq 2$, assume by induction that we already have a labeling of the vertices of $G_1 \oplus \dots \oplus G_{\ell-1}$. Let $A_\ell$ be the apex set of $G_\ell$ and $C_\ell$ be the clique involved in the clique-sum of $(G_1 \oplus \dots \oplus G_{\ell-1}) \oplus G_\ell$. Recall that $C_\ell \subseteq A_\ell$. Obtain a labeling by

- using Grohe's Theorem on $G_\ell - A_\ell$
- letting $C_\ell$ inherit its labels from $G_1 \oplus \dots \oplus G_{\ell-1}$
- choosing arbitrary labels for $A_\ell - C_\ell$

Let $D$ be the vertices of the label we want to delete. Note that since $C_\ell - D$ is still a clique, we have:




$$\left.\begin{aligned}&(G_1 \oplus \dots \oplus G_\ell) - D \\ &= \underbrace{\big((G_1 \oplus \dots \oplus G_\ell) - D\big)}_{\substack{tw \,\leq\, C_{tf} \cdot k \\ \text{by induction hypothesis}}} \oplus \underbrace{(G_\ell - D)}_{\substack{tw \,\leq\, C_{tf} \cdot k \\ \text{by construction}}}\end{aligned}\right\} \; (*)$$

and hence $tw(G - D) \leq C_{tf} \cdot k$.

In order to obtain an edge-labeling, proceed as follows:

- for an edge on $G_\ell - A_\ell$, let it have the label of its endpoint which is closest to the root of the BFS-tree of $G_\ell$.
- for an edge in $C_\ell$, let it inherit its label from $G_1 \oplus \dots \oplus G_{\ell-1}$
- for an edge in $G_\ell$ with <u>exactly one</u> endpoint $u$ in $C_\ell$, let it have the label of $u$.
- for an edge with at least one endpoint $u$ in $A_\ell - C_\ell$, let it have the label of $u$, break ties arbitrarily.

If $D$ now denotes the set of edges of the deleted label, it is not hard to check that $(*)$ still holds.

If $u$ has the deleted label, remove it from $G_\ell$ and $C_\ell$.

Some ideas for the proof of Grohe's theorem:

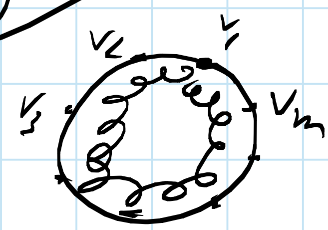
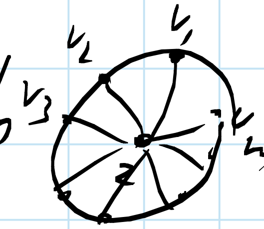
Let $G, H$ be graphs with $V(G) \cap V(H) = \{v_1, \ldots, v_m\}$.

Assume $H$ has a path decomposition of width $k$ as follows:


That is, $v_i \in B_i$. Then we have $tw(G \cup H) \leq (tw(G)+1)(k+1)-1$.

Idea: Consider a tree decomposition $(T, (C_t)_{t \in T})$ of $G$.

Define $C'_t = C_t \cup \bigcup_{\substack{1 \leq i \leq m \\ v_i \in C_t}} B_i$. Then $(T, (C'_t)_{t \in T})$ is the desired tree decomposition of $G \cup H$.


Now replace each vortex  with 

→ The graph becomes bounded-genus → has linear local treewidth. Now "paste" the path decomp. of the vortex using the observation above.

Handling minors is more complicated. Need to argue that shortest path and hence neighborhoods increase only proportional to $h$ (and not $n$) when contracting edges → see [Grohe '03].

## References

[DH04] Erik D. Demaine and MohammadTaghi Hajiaghayi. Equivalence of local treewidth and linear local treewidth and its algorithmic applications. In *SODA '04: Proceedings of the 15th annual ACM-SIAM Symposium on Discrete Algorithms*, pages 840–849, Philadelphia, PA, USA, 2004. Society for Industrial and Applied Mathematics.

[DHK05] Erik D. Demaine, MohammadTaghi Hajiaghayi, and Ken-ichi Kawarabayashi. Algorithmic graph minor theory: Decomposition, approximation, and coloring. In *FOCS '05: Proceedings of the 46th annual IEEE Symposium on Foundations of Computer Science*, pages 637–646, Washington, DC, USA, 2005. IEEE Computer Society.

[Epp00] David Eppstein. Diameter and treewidth in minor-closed graph families. *Algorithmica*, 27(3):275–291, 2000.

[Gro03] Martin Grohe. Local tree-width, excluded minors, and approximation algorithms. *Combinatorica*, 23(4):613–632, 2003.

[KW11] Ken-ichi Kawarabayashi and Paul Wollan. A simpler algorithm and shorter proof for the graph minor decomposition. In *STOC'11: Proceedings of the 43rd ACM Symposium on Theory of Computing*, pages 451–458, 2011.