6.889	Lecture 6	Sept. 28, 2011

Small Separators in H-minor-free Graphs

Recall separation in G:

- partition (A, B, X) of $V(G)$
- no edge between A and B
- $|X|$ is the order
- balanced if $w(A), w(B) \leq \frac{2}{3} w(V(G))$

Lipton-Tarjan: balanced separator of order $\leq \sqrt{8n}$ in planar graphs. '79

→ linear time.

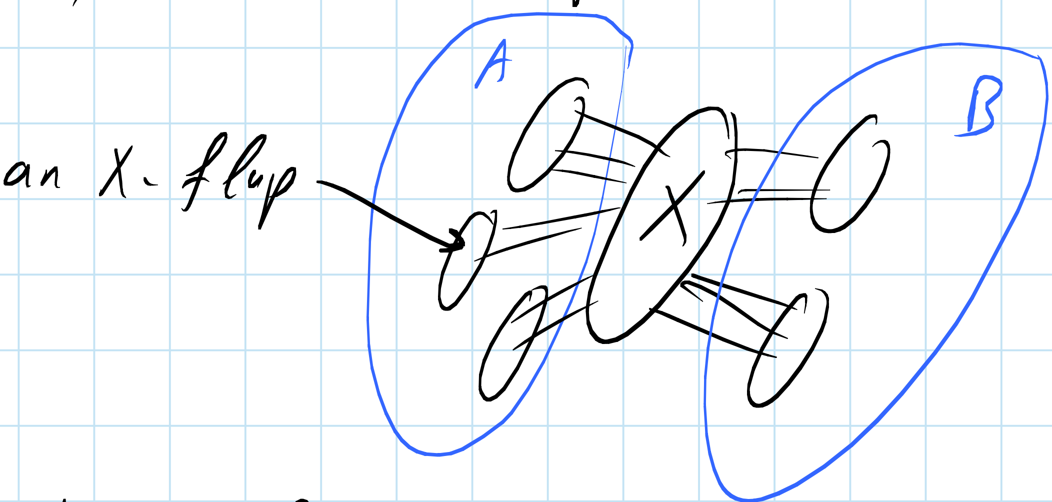
Alon-Seymour-Thomas (this lecture): balanced separator of order $\leq h^{3/2}\sqrt{n}$ in H-minor-free graphs. $(|H| = h)$ '90

→ time $O(h^{1/2} n^{3/2})$.

Reed-Wood: - linear time but order $O(n^{2/3})$ '05
- use AST with knitted H-partition
- exponential dependence on h

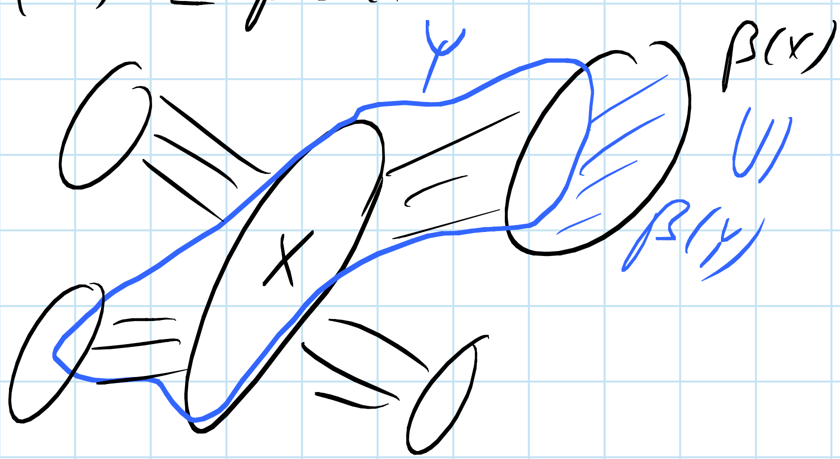
Wulff-Nielsen: - order $O(n^{1-\varepsilon})$ in linear time '11
and polynomial dependence on h

For $X \subseteq V(G)$, an X-flap is the vertex set of any connected component of $G - X$.



Notice: If for every X-flap F, we have $w(F) \leq \frac{1}{2} w(G)$, we easily obtain balanced separation.

A haven of order k is a function β that assigns to each subset $X \subseteq V(G)$ with $|X| \leq k$ an X-flap $\beta(X)$, in such a way that if $X \subseteq Y$ and $|Y| \leq k$, then $\beta(Y) \subseteq \beta(X)$.



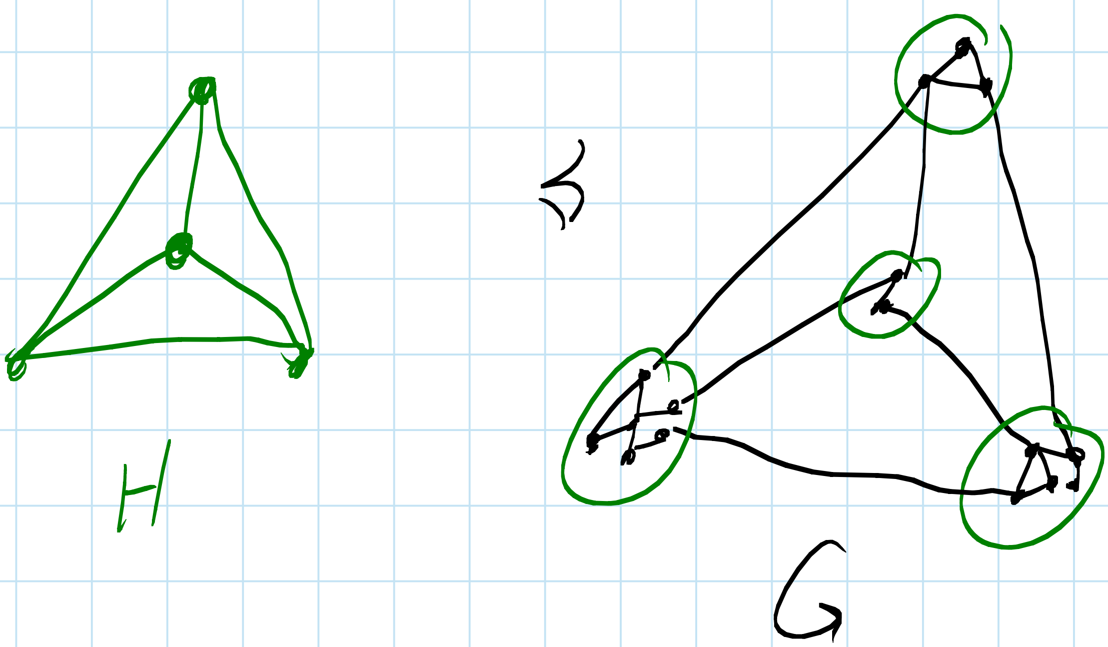
Suppose there is no X of size $\leq h^{3/2}\sqrt{n}$ such that every X-flap F has weight $\leq \frac{1}{2} w(G)$

$\Rightarrow$ define haven of order $h^{3/2}\sqrt{n}$, where $\beta(X)$ is the unique X-flap of weight $> \frac{1}{2} w(G)$.

Hence, if G has <u>no</u> haven of order $h^{3/2}\sqrt{n}$, then we have balanced separations as desired.

Main Theorem: Let $h \geq 1$ be an integer, and let G be a graph on n vertices with a haven of order $h^{3/2}\sqrt{n}$. Then G has a K_h-minor.

Recall: H is minor of G $\Leftrightarrow$ G contains model of H



Green circles are connected subgraphs.

Lemma on small connecting trees

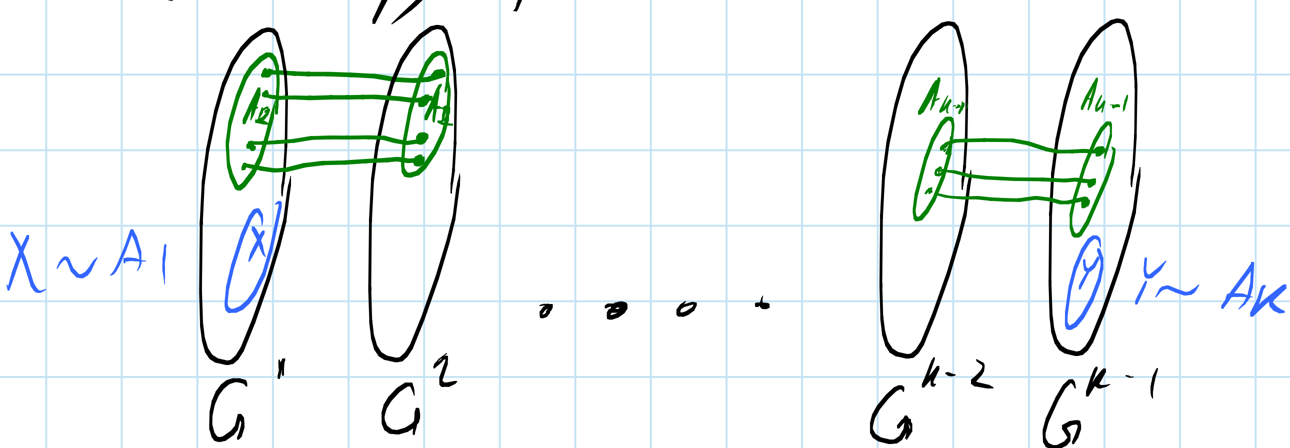
Given graph G, real number $r \geq 1$, and k subsets $A_1, \dots, A_k \subseteq V(G)$, either

(i) there is a tree T in G of size $\leq r$ that intersects every A_i, that is $V(T) \cap A_i \neq \emptyset$, for all $1 \leq i \leq k$.

(ii) or there exists $Z \subseteq V(G)$ of size $\leq (k-1)n/r$, such that no Z-flap intersects all of $A_1, \dots, A_k$.

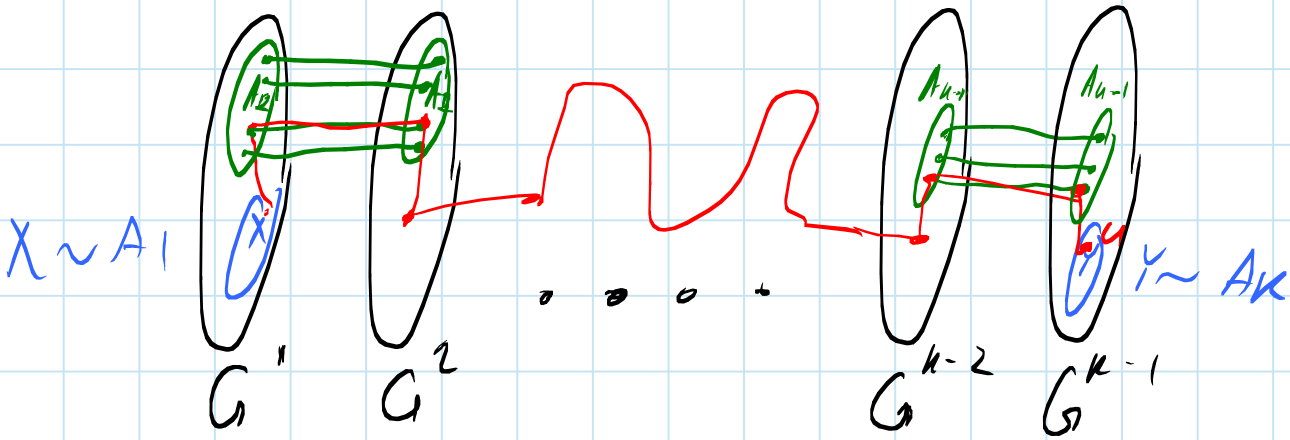
[In other words, either there is a small tree that connects all of the A_i or a small separator that "separates at least one of them from the rest"]

Proof: Let $G^1, \dots, G^{k-1}$ be isomorphic copies of G. For each A_i with $2 \leq i \leq k-1$, add a matching between copies of A_i in G^{i-1} and G^i. Call this big graph J. Let X be copy of A_1 in G^1 and let Y be copy of A_k in G^{k-1}.



For each $u \in V(J)$, let $d(u) :=$ # vertices on shortest path from X to u or ∞ if no such path exists.

Case (i): $d(u) \le r$ for some $u \in Y$.



Red path corresponds to a tree of size $\le r$ in G
$\longrightarrow$ done. ✓

Case (ii): $d(u) > r$ for all $u \in Y$.

Let t be smallest integer with $t \ge r$. For $1 \le j \le t$, let

$Z_j = \{u \in V(J) : d(u) = j\}$.

Note that $Z_1, \dots, Z_t$ are disjoint and $|V(J)| = (k-1)n$
$\Rightarrow \exists\, Z_j$ of cardinality $\le (k-1)n/t \le (k-1)n/r$.

Let Z be the vertices of G corresponding to Z_j.

Claim: Z is our desired separator.

Note that every path between X and Y in J has a vertex in Z_j, since $d(u) \geq j$ for all $u \in Y$.

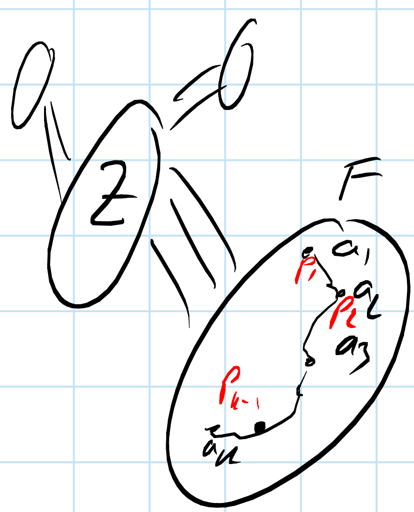
Suppose there is a Z-flap F that intersects all of $A_1, \dots, A_k$. For $i = 1, \dots, k$ pick $a_i \in A_i \cap F$. Construct a path $P \subseteq F$, where

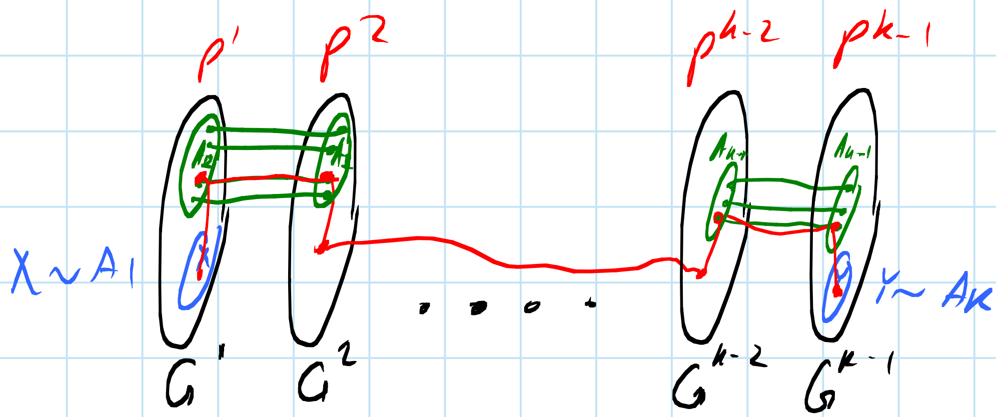
$$P : a_1 \underset{P_1}{\sim} a_2 \underset{P_2}{\sim} a_3 \dots \sim a_{k-1} \underset{P_{k-1}}{\sim} a_k$$

Let P^i be the path in G^i corresponding to P_i.

Let $Q : P^1 \sim P^2 \sim \dots \sim P^{k-1} \subseteq J$.

Then Q is a path between X and Y in J that avoids Z_j $\longrightarrow$ contradiction. ✓





$\blacksquare$

Proof of main theorem:

Let β be a haven in G of order $h^{3/2}\sqrt{n}$.
Have to show that G contains K_h-minor.

Choose $X \subseteq V(G)$, $k \leq h$, and a K_k-model $\mathcal{C} = \{C_1, \dots, C_k\}$ in G, such that

(i) $X \subseteq \bigcup_{C \in \mathcal{C}} V(C)$

(ii) $|X \cap V(C)| \leq \sqrt{hn}$ for each $C \in \mathcal{C}$

(iii) $V(C) \cap \beta(X) = \emptyset$ for each $C \in \mathcal{C}$

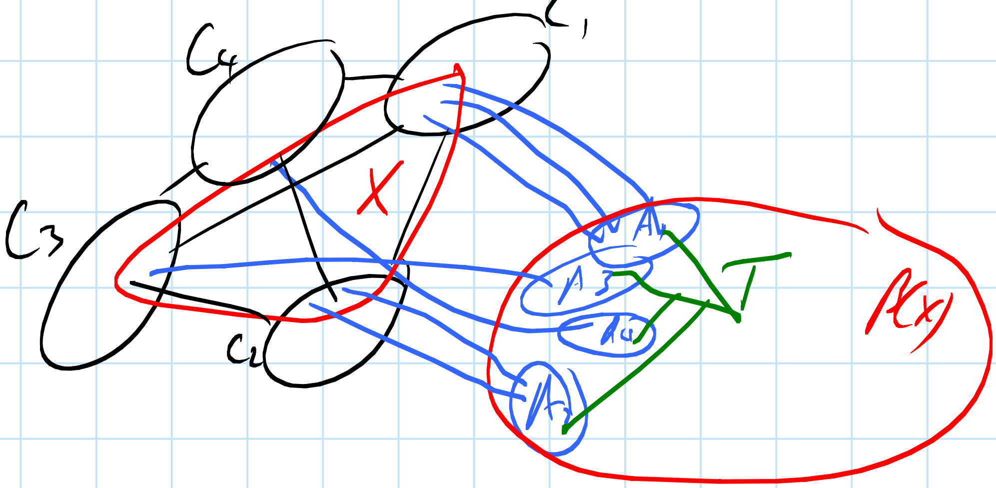
(iv) subject to (i), (ii), (iii), the term

$$k + 2(|\beta(X)| + |X \cup \beta(X)|)$$ is minimized.

This certainly possible since $X = \mathcal{C} = \emptyset$ and $k = 0$ satisfy (i), (ii), and (iii).

Let $A_i := \{ v \in \beta(X) \mid v \text{ is adjacent in } G \text{ to some } u \in C_i \}$

Let $G' := G[\beta(X)]$, $r = \sqrt{hn}$ and apply Lemma!

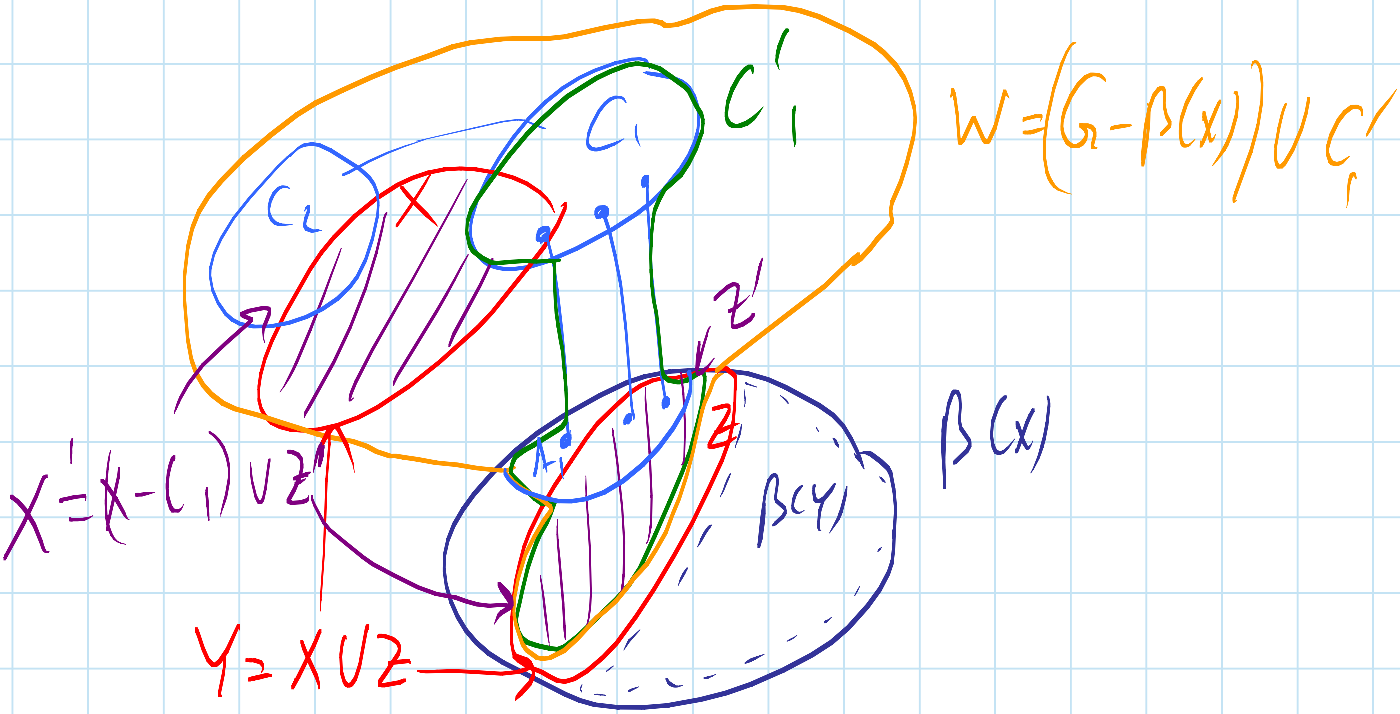


Case 1: There is a small tree T of size $\leq \sqrt{hn}$ that intersects all of $A_1, \dots, A_k$.

Let $\mathcal{C}' = \mathcal{C} \cup \{T\}$, $X' = X \cup V(T)$.

$\longrightarrow$ contradiction to (iv) ✓

Case 2: There is $Z \subseteq \beta(X)$ of size $\leq \dfrac{(k-1)|\beta(X)|}{\sqrt{hn}}$ such that no Z-flap of G' intersects all of $A_1, \dots, A_k$.



Define $Y = X \cup Z \Rightarrow |Y| \le h^{3/2}\sqrt{n}$

$\Rightarrow \beta(Y)$ exists and $\beta(Y) \subseteq \beta(X)$.

$\beta(Y)$ is Z-flap of $G' \Rightarrow \exists\, i$ s.t. $\beta(Y) \cap A_i = \emptyset$

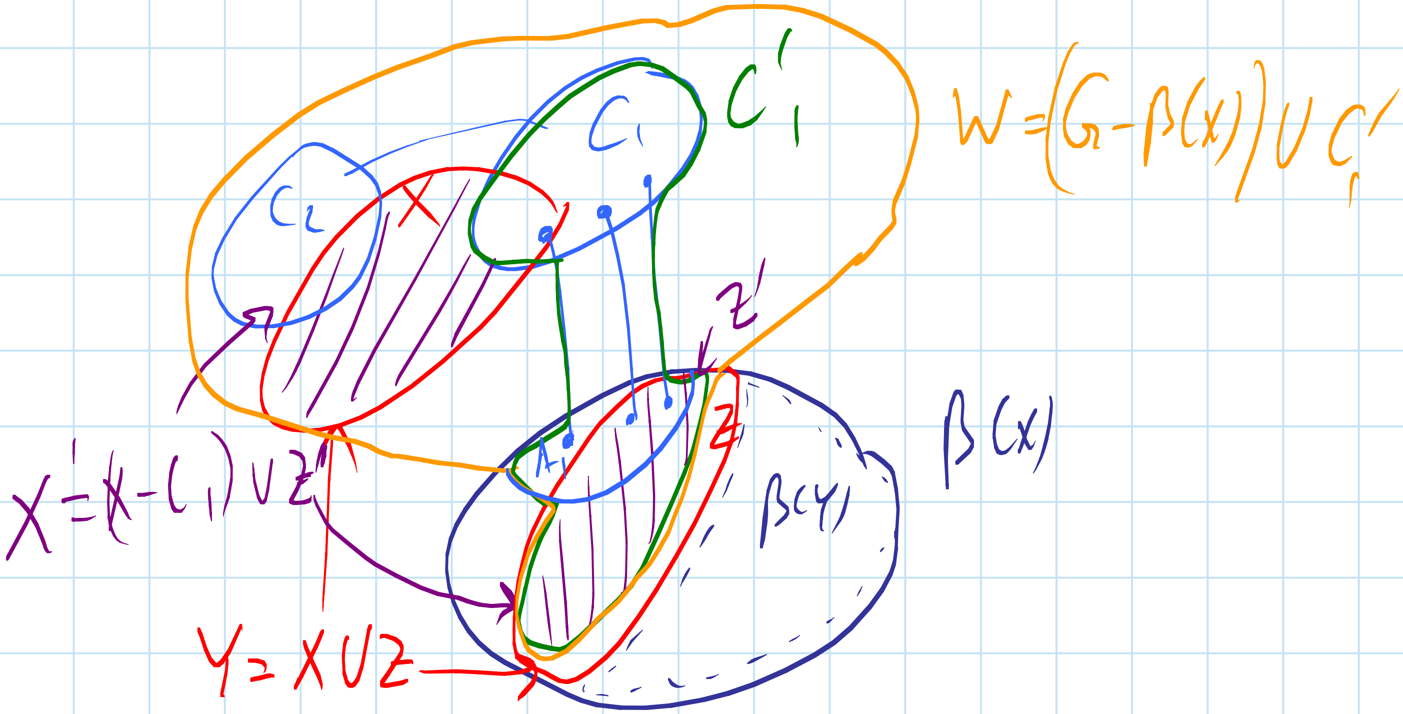
w.l.o.g. $i=1$.

$\boxed{\text{→ extend } C_1 \text{ to a maximal tree } C_1' \text{ in } G \text{ disjoint from } \beta(Y) \text{ and } C_2, \dots, C_k}$

Let $Z' = V(C_1') \cap Z$, $X' = Z' \cup (X - V(C_1)) \subseteq Y$, and $W = (V(G) - \beta(X)) \cup V(C_1')$.

Claim: $\beta(X') \cap W = \emptyset$ and

hence $\beta(X') \subseteq \beta(X)$.



Assume claim is true. Let $\mathcal{C}' = \mathcal{C} - \{C_1\} \cup \{C_1'\}$.

(i) $X' \subseteq \bigcup_{C \in \mathcal{C}'} V(C)$ since $Z' \subseteq V(C_1')$

(ii) $|X' \cap V(C)| \leq \sqrt{hn}$ for each $C \in \mathcal{C}'$ since $X' \cap V(C_1') = Z'$

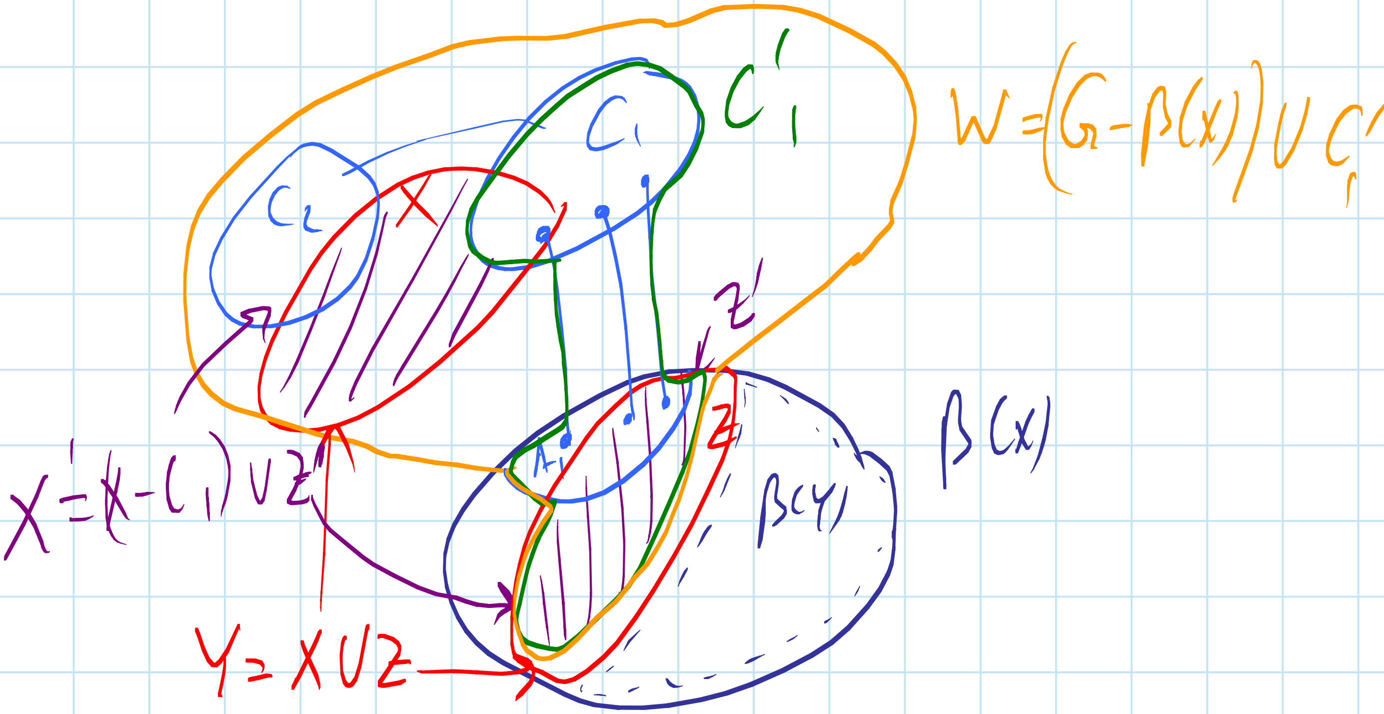
(iii) $V(C) \cap \beta(X') = \emptyset$ for each $C \in \mathcal{C}'$ since $\beta(X') \cap W = \emptyset$

By (iv), we have $|\mathcal{C}'| + 2(|\beta(X')| + |X' \cup \beta(X')|) \geq |\mathcal{C}| + 2(|\beta(X)| + |X \cup \beta(X)|)$

but $|\mathcal{C}'| = |\mathcal{C}|$, $\beta(X') \subseteq \beta(X)$, and $X' \cup \beta(X') \subseteq (X \cup \beta(X)) - (X \cap V(C_1))$

$\Rightarrow X \cap V(C_1) = \emptyset \Rightarrow \mathcal{C} - \{C_1\}, X$ satisfy (i), (ii), and (iii) contrary to (iv)!

$\longrightarrow$ contradiction ✓



Remains to show claim: $\beta(X')\cap W=\emptyset$.

Suppose not $\Rightarrow \exists$ path P in G between W and $\beta(Y)$ since $\beta(Y)\subseteq\beta(X')$ $(X'\subseteq Y)$ and $\beta(X')$ is connected.

$\Rightarrow P$ disjoint from X'.

Since $W\cap\beta(Y)=\emptyset \Rightarrow \exists\, uv\in P$ with $u\in W$, $v\in G-W\subseteq\beta(X)$

$\Rightarrow u\in X\cup\beta(X) \Rightarrow u\in (X\cup\beta(X))\cap(W-X')\subseteq V(C_1')$

$v\notin W$, hence maximality of C' implies $v\in\beta(Y) \xRightarrow{u\notin\beta(Y)} u\in Y$

$\Rightarrow u\in Y\cap(V(C_1')-X')\subseteq V(C_1)$.

$\Rightarrow v\in A_1$ and $v\in\beta(Y)$ but $A_1\cap\beta(Y)=\emptyset$ by Lemma

$\longrightarrow$ contradiction ✓

∎

References

[AST90] Noga Alon, Paul Seymour, and Robin Thomas. A separator theorem for nonplanar graphs. *Journal of the American Mathematical Society*, 3(4):801–808, 1990.

[RW09] Bruce Reed and David R. Wood. A linear-time algorithm to find a separator in a graph excluding a minor. *ACM Transactions on Algorithms*, 5(4):1–16, 2009.