6.889	Lecture 5	Sept. 26, 2011

Nonnegative SSSP in planar graphs:

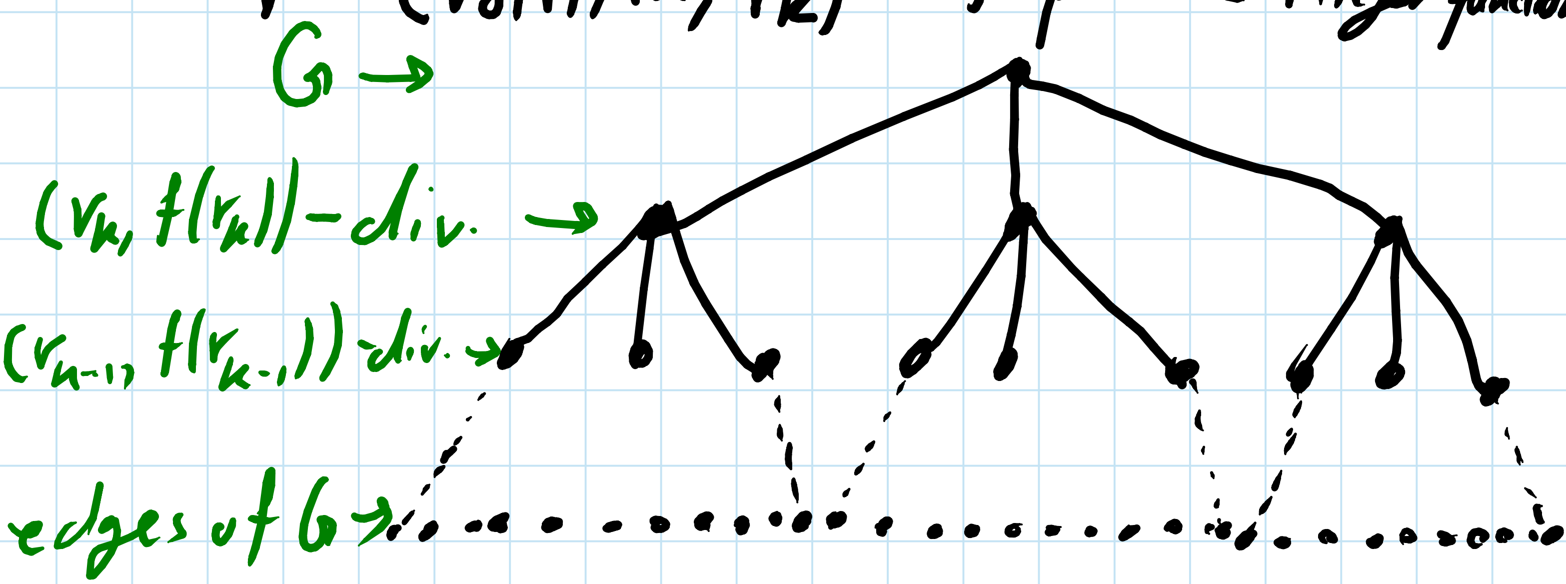
2-level division $\longrightarrow$ $O(n \log \log n)$
recursive division $\longrightarrow$ $O(n)$

(r, s)-division: partition edges of G into

- $O(n/r)$ regions
- each containing $r^{O(1)}$ vertices → relaxed cond.
- and at most s boundary vertices

$(\bar{r}, f)$-recursive division:

$\bar{r} = (r_0, r_1, \dots, r_k)$ f: positive integer function



Theorem [HKRS '97]:

Given graph G with max. in/out-degree 2 and $(\bar{r}, cf)$-recursive division tree that satisfies

$$\frac{r_i}{f(r_i)} \geq 8^i f(r_{i-1}) \log r_{i+1} \left(\sum_{j=1}^{i+1} \log r_j \right) \quad (*)$$

for all r_i exceeding a constant,

the nonnegative SSSP on G can be solved in linear time.

A suitable linear-time recursive division can be constructed for any class that is

- minor-closed
- admits small separators $\rightarrow O(n^{1-\varepsilon})$
  in linear time

Catch: algorithm also requires bounded degree.

$\hookrightarrow$ uses ρ-clustering

Theorem [RW '09 based on AST '90]:
Every proper minor-closed class of graphs admits a balanced separator of size $O(n^{2/3})$ that can be found in linear time.

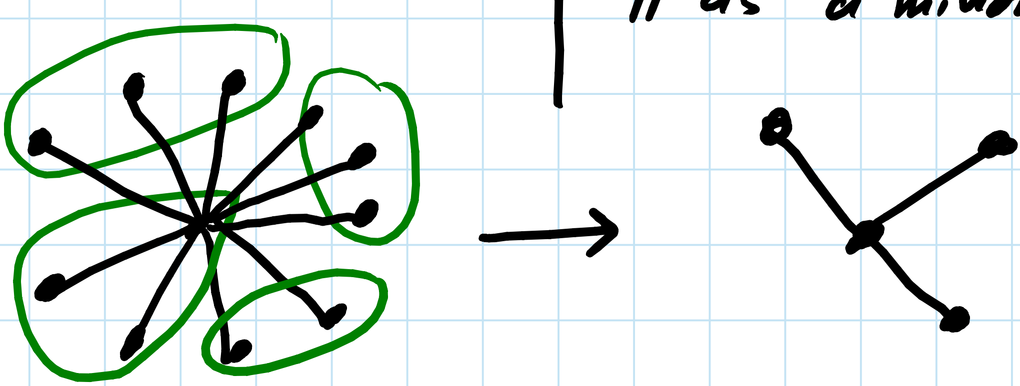
$\longrightarrow$ next lecture

What about <u>bounded degree</u> requirement?
- easy for surface-embedded graphs
- <u>not possible</u> for arbitrary H-minor-free graphs!

Claim: For every given $k \in \mathbb{N}$, there exists a K_6-minor-free graph G_k that contains a vertex v such that for any permutation π_v, the graph resulting from <u>splitting</u> v according to π_v contains K_k as a minor.

$\longrightarrow$ see exercises!

ρ-clustering / H-model	knitted H-Partition
disjoint partition $\mathcal{P}=\{P_1,\dots,P_t\}$ of $V(G)$	disjoint partition $\mathcal{P}=\{P_1,\dots,P_t\}$ of $V(G)$
$t=O(n/\rho)$	$t=O(n/\rho)$
$\lvert P_i\rvert=\Theta(\rho)$	$\lvert P_i\rvert=\Theta(\rho)$
each part is connected	
bijection $\pi: V(H)\to\mathcal{P}$	bijection $\pi: V(H)\to\mathcal{P}$
$\forall uv\in E(H)$, there is an edge between $\pi(u)$ and $\pi(v)$ in G $\Longrightarrow$	$\forall uv\in E(H)$, there is an edge between every connected component of $\pi(u)$ and every connected component of $\pi(v)$ in G $\Longrightarrow$
contracting the parts gives H as a minor	collapsing each part gives H as a minor



Theorem [Reed-Wood '09]:

There is a linear-time algorithm that, given a constant z and a graph G excluding K_ℓ as a minor, outputs a knitted H-partition $\mathcal{P} = \{P_1, \dots, P_t\}$ of G such that $t \leq n/z$ and $|P_i| < c_0 \cdot z$ for all $1 \leq i \leq t$, where $c_0 = 2^{\ell^2} + 1$.

↳ next lecture

Theorem [T.-Müller-Hannemann '09]:

There is a linear time algorithm that, given a K_ℓ-minor-free graph G, finds an $(\bar{r}, f)$-recursive division of G that satisfies inequality (*) for all r_i exceeding a constant.

Corollary: In every H-minor-free class of graphs, single-source shortest paths can be computed in linear time.

Generalized Recursive Division Algorithm:

Input: K_ℓ-minor-free graph G

// shrink the graph recursively

let $n := |V|$, $G_0 := G$, $z_0 := 2$, $i := 0$;

while $|V(G_i)| > \frac{n}{\log n}$ do

let $G_{i+1} := \text{H-Partition}(G_i, z_i, \ell)$;

let $z_{i+1} := 14^{z_i^{1/7}}$, $i := i+1$;

let $I := i-1$,

// build r-divisions in small graphs and expand back to bigger graphs to obtain recursive division tree T

let v_G be the root of T;

let D_{I+1} be the trivial division of G_{I+1} with single region

for $i := I$ downto 0 do

for each region R of D_{i+1} do

let S_R be the boundary vertices of R;

let $D_R := \text{Divide}(R, S_R, z_i, \ell)$;

for each region R' of D_R do

expand R' into region R'' of G_i by expanding every vertex; assign each boundary edge to a region; create a child $v_{R''}$ of v_R in T.

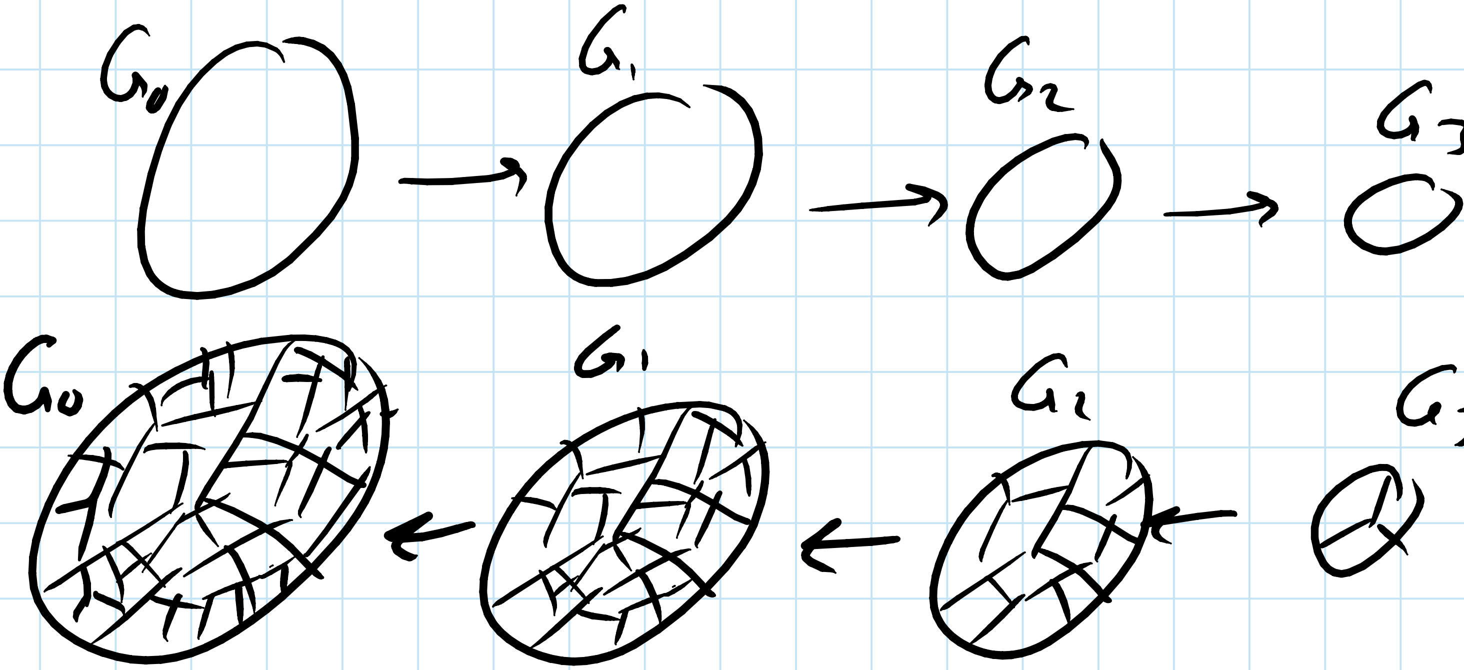
let D_i be the decomposition of G_i consisting of the regions R'' above;

// add the leaves

Divide (G, S, r, ℓ):

basically an r-division using small separators of size $O(n^{2/3})$, achieving:

- at most $O(|S|/r^{2/3} + \frac{n}{r})$ regions
- each region at most r vertices
- each region at most $O(r^{2/3})$ boundary vertices where all vertices of S count as boundary
- takes time $O(n \log n)$



Overview of analysis:

- each region of D_i has at most $O(z_i^2)$ vertices and at most $O(z_i^{5/3})$ boundary vertices.

- show number of regions k_i in division D_i is $O(n_i/z_i^2)$ where $n_i = |V(G_i)|$ (reverse induction on i)
- show algorithm runs in linear time:
  - time to form graphs $G_1, \ldots, G_{I+1}$: $O(\sum_i n/z_i) = O(n)$
  - time to apply Divide to region R of G_{i+1} with n_R vertices:
    $O(n_R \log n_R) = O(n_R \log z_{i+1})$,
    hence $\sum_R O(n_R \log z_{i+1}) = O(n_{i+1} \log z_{i+1})$
  - time to obtain divisions of all the G_i:

$$\sum_i O(n_{i+1} \log z_{i+1}) = \sum_i O\left(\frac{n}{z_i} \cdot z_i^{1/7}\right) = O(n)$$

since $n_{i+1} \leq n_i / z_i \leq n/z_i$

and $\log z_{i+1} = O(z_i^{1/7})$

- show inequality (*) holds (see paper)

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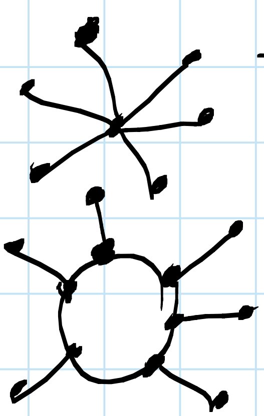
still need to achieve bounded degree for SSSP



- fix an in-order traversal of recursive division tree

$\rightarrow$ induces a total circular order on all edges

- split each vertex and order its incident edges according to this order.

References

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[RW09] Bruce Reed and David R. Wood. A linear-time algorithm to find a separator in a graph excluding a minor. *ACM Transactions on Algorithms*, 5(4):1–16, 2009.

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