Characterization of flat-foldable (single-vertex) mountain-valley patterns

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- number of mountains − number of valleys = ±2 [Maekawa; Justin 1986]
  - measure total turn angle = 180° − interior angle
    (> 0 for convex vertices
    < 0 for reflex vertices)
    - ↳ turn along arcs cancels out (equal cw & ccw — no winding)
    - ↳ each crease turns +180° for mountain
      or −180° for valley
  - no crossing ⇒ total turn angle = ±360°.

  ⇒ 180°(# mountains − # valleys) = ±360°
  ⇒ # mountains − # valleys = ±2. □

- strict local min. angle is surrounded by one mountain & one valley [Kawasaki 1989; Justin 1984]

⇒ can crimp immediately any such angle (preserves flat foldability as before)

- consider k equal angles surrounded by strictly larger angles
  then #mountains − #valleys = {0 if k is odd; ±1 if k is even} [Hull 2001 & 2003]
  (among creases between these k+2 angles)
  - k even ⇒ extend one of larger angles to match the other & close into cycle of angles

  - k odd ⇒ add new angle = Σ larger angles − equal angle adjacent to both larger angles

  - flat foldable, with same M-V assignment on these creases as original
  - k even ⇒ 180° turn angle + new crease ⇒ M − V = ±1
  - k odd ⇒ 0 turn angle + 2 new creases to "circle" ⇒ M − V = 0. □

∴ there is at least one crimp among these creases (use globally min. angle)
(unless all angles are equal, in which case M − V = ±2 ⇒ ∄ crimp)

- M-V pattern flat foldable ⇔ CP is flat foldable & there is a crimp whose result is flat foldable
  (base case: 2 angles ⇒ both M or both V)
- algorithm to count # valid M-V assignments [Hull 2003]
  (linear-time)
  - all equal angles: $2\binom{n}{n/2-1}$ (max.) — similar within indep. groups of equal angles (min. $= 2^{n/2}$)