6.885
L6.2
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Definitions: [Demaine, Devadoss, Mitchell, O'Rourke 2004; Lang 2004]

Piece of paper = connected (metrizable) manifold
- usually: 1D line segment (0, 1) OR 2D polygon

↳ always the interior
↳ optionally the boundary (open or closed set)

Folded-state geometry of d-D paper P = function $f: P \to \mathbb{R}^{d+1}$, such that f is **isometric** = preserve intrinsic distances

↳ in 1D: $d_P(p,q) = |p-q|$
$d_f(p,q) = \text{arclength}_{[p,q]} f = \left| \int_p^q \left\| \frac{d}{dx} f(x) \right\| dx \right|$.

↳ in 2D+: $d_P(p,q) = \inf\{\text{arclength } C \mid \text{curve } C:[0,1] \to P$ starting at $C(0)=p$ & ending at $C(1)=q\}$
$d_f(p,q) = \inf\{\text{arclength}(f \circ C) \mid \text{curve } C \text{ from } p \text{ to } q\}$

↳ $d_P(p,q) = d_f(p,q)$ for all $p,q \in P$

Free folded state = one-to-one folded-state geometry
- no touching ⇒ no crossing

Piecewise-C^k = finite number of (d-1)-D submanifolds (of P) ↳ **creases** between which f is C^k i.e. has continuous k^{th} derivatives
- assume piecewise-C^1

Folded-state order = partial function $\lambda: P^2 \dashrightarrow \{+1, -1\}$.
- $\lambda(p,q)$ defined ⇔ p & q in contact i.e. (connected) neighborhoods N_p & N_q (containing no crease points) with $f(N_p) = f(N_q)$ & p & q are noncrease (nonboundary) points
- $\lambda(p,q)$ measures whether p is "above" (+1) or "below" (−1) q in folded state, relative to **normal** of f at q
  ↳ defines "top side" of paper

- **consistency:** connected contacting nhbrhds N_p & N_q ⇒ $\lambda(\tilde{p},\tilde{q}) = \lambda(p,q)$ $\forall \tilde{p} \in N_p, \tilde{q} \in N_q$
- **symmetry:** normal $_p = \pm$ normal $_q$ ⇒ $\lambda(p,q) = \mp \lambda(q,p)$
- **transitivity:** $\lambda(p,q) = -\lambda(r,q)$ ⇒ $\lambda(p,r) = \lambda(q,p)$