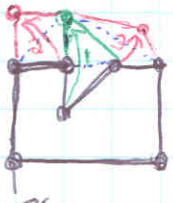
Flipping pockets

6.885
L5.4
Sept. 22, 2004

- **pocket** of 2D polygon = region of convex hull minus the polygon
- **pocket lid** = edge of convex hull incident to pocket
- **pocket flip** = rotate pocket (chain of polygon incident to pocket) 180° through 3D around the pocket lid.
  - preserves non-self-intersection of polygon
  - increases area
  - convexifies? [Erdős 1935]

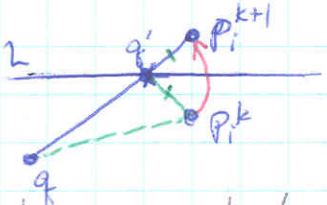


3 flips

Erdős-Nagy Theorem: [Nagy 1939] [Toussaint 1999]
polygon convexifies after a finite number of flips.

Proof:

- let $P^k = p_1^k p_2^k \cdots p_n^k$ be the polygon after k flips.
- for any point q in P^k, $\|q - p_i^k\| \le \|q - p_i^{k+1}\|$
  - q is in P^{k+1} (containment)
  - if p_i unflipped @ $k+1^{ST}$ flip, holds with equality
  - if p_i flipped about pocket lid L = perp. bisector of p_i^k & p_i^{k+1} then
$$d(q, p_i^{k+1}) = d(q, q') + d(q', p_i^{k+1}) = d(q, q') + d(q', p_i^k) \ge d(q, p_i^k)$$



- fix q in P^0

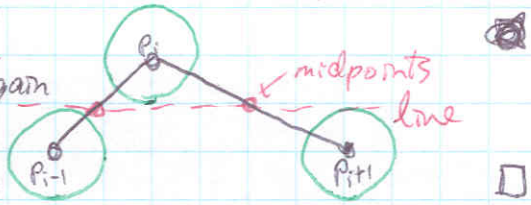
$\Rightarrow \|q - p_i^k\|$ monotonic ↗ in k & at most perimeter $\Rightarrow$ has unique limit

- repeat for 3 points q in P^0 $\Rightarrow$ p_i^k converges to unique limit p_i^*.
- limit polygon $P^* = p_1^* p_2^* \cdots p_n^*$
- non-crossing by expansiveness: if p_i & p_j flip together, dist. constant.
  if p_i flips & p_j doesn't, use $q = p_j^k$ $\Rightarrow \|p_i^k - q\| \le \|p_i^{k+1} - q\|$.

$\Rightarrow$ convex because otherwise could flip more ← **BUG!** (see L7.1)

- draw small disk around each p_i^* (with angle $< 180°$) such that perturbing within disks leaves convex all vertices
- by limit, after $k \ge c$, each p_i^k is in p_i^*'s disk (& remains there).
- then p_i^k is convex. if angle $< 180°$
- line separator $\Rightarrow$ p_i^k never flipped again $\Rightarrow p_i^k = p_i^*$

$\Rightarrow$ ditto for $180°$-angle vertices (dist.)



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