Laman's characterization [Laman 1970]

6.885
L3.5
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G is minimally gen. rigid in 2D
$\Leftrightarrow |E| = 2n-3$
and every (induced) subgraph S of G has $|E(S)| \le 2|V(S)|-3$.

Proof: ($\Rightarrow$) - easy by Henneberg characterization:
- induct on operations to build G (i.e. on n)
- each op. adds one vertex v
  & each subgraph S excludes $v \Rightarrow$ no change
  or includes $v \Rightarrow |V(S)| \nearrow 1$
  $|E(S)| \nearrow \le 2$.

(& base case of ⟷)

($\Leftarrow$) $|E| = 2n-3$ so as before there is a vertex of degree 2 or 3

degree 2: - apply anti-Type I (remove v)
- every subgraph of resulting G' is a subgraph of the original G
- $\Rightarrow$ still Laman
- induct

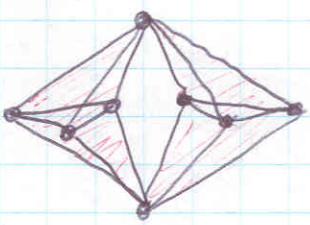
degree 3: some edge is missing among neighbors (of v) because induced graph on 4 vertices has $\le 2 \cdot 4 - 3 = 5$ out of 6 edges.
$\Rightarrow$ can apply anti-Type II.
with more argument, can show there is an anti-Type II preserving Laman induct

□

Somewhat involved - see Counting on Frameworks by Jack E. Graver

OPEN: characterize (min.) gen. rigid graphs in 3D

- Laman condition (with $2k-3 \to 3k-6$) is still necessary for min. gen. rigidity
- insufficient:



"double bananas"