6.885
L21.6
Nov. 29, 2004

Characterization: $r \in \mathbb{R}$ constructible via origami
$\Leftrightarrow$ there exist $\alpha_1, \alpha_2, \ldots, \alpha_n = r \in \mathbb{R}$
such that α_i is a root of a degree-2 or -3 polynomial over $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{i-1})$, for all i

Proof sketch:
- again constructible numbers form field (can subtract & divide)

($\Rightarrow$) each operation solves a particular degree-2 or -3 polynomial
- only Axiom 6 requires a degree-3 polynomial
- Intuition: common tangent to two parabolas (degree-2 curves) leads to two degree-2 polynomials which can be combined into one degree-4 polynom. $= 2 \cdot 2$
- degree-4 polynomial reduces to degree-2 or -3 polynomial [Euler]

($\Leftarrow$) Cubic formula: [? del Ferro ~1500]

$x^3 + a_2 x^2 + a_1 x + a_0 = 0$
has roots $-\frac{1}{3}a_2 + (S+T)$,
$-\frac{1}{3}a_2 - \frac{1}{2}(S+T) + \frac{1}{2} i\sqrt{3}(S-T)$,
$-\frac{1}{3}a_2 - \frac{1}{2}(S+T) - \frac{1}{2} i\sqrt{3}(S-T)$,

where $S = \sqrt[3]{R+\sqrt{D}}$
$T = \sqrt[3]{R-\sqrt{D}}$
$D = Q^3 + R^2$ (discriminant)
$Q = \frac{1}{3}a_1 - \frac{1}{9}a_2^2$
$R = \frac{1}{6}a_1 a_2 - \frac{1}{2}a_0 - \frac{1}{27}a_2^3$

- suggests we only need field + square roots + cube roots
- but: can't produce all roots this way without going thru $\mathbb{C}$!
(even when the roots are all real & distinct — $D<0$)
- better characterization of roots:

$2\sqrt{-Q}\cos(\frac{1}{3}\theta) - \frac{1}{3}a_2$
$2\sqrt{-Q}\cos(\frac{1}{3}\theta + 120°) - \frac{1}{3}a_2$
$2\sqrt{-Q}\cos(\frac{1}{3}\theta + 240°) - \frac{1}{3}a_2$

where $\cos\theta = R/\sqrt{-Q^3}$.

$\Rightarrow$ suffices to have field, square root, cube root, and angle trisection!

($\Leftarrow$) known in [Piazzolla Beloch 1936], [Sustin 1989], [Geretschläger 1995]
($\Rightarrow$) proved in [Huzita & Scimemi 1989], [Alperin 2000]