Origami constructible numbers

6.885
L21.1
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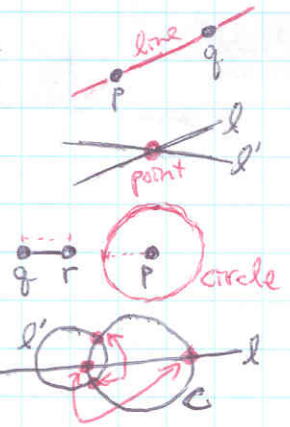
Straight edge & compass ["collapsing"] (for comparison)

AXIOMS:
1. given points p, q, construct line thru p & q
2. given (nonparallel) lines l, l', construct point $l \cap l'$
3. given points p, q, r, construct circle centered at p with radius $\|q-r\|$
4. given circle C & line or circle l, construct $C \cap l$ (≤ 2 points)



Constructible numbers = constructible coordinates starting from two points $(0,0)$ & $(1,0)$

Field extension $F(\alpha)$ = smallest field containing field F & number α
$= \left\{ \frac{f(\alpha)}{g(\alpha)} \mid f, g \text{ are polynomials in } F \text{ \& } g(\alpha) \neq 0 \right\}$

Characterization: $r \in \mathbb{R}$ is constructible via SE&C
$\Leftrightarrow$ there exist $\alpha_1, \alpha_2, \ldots, \alpha_n = r \in \mathbb{R}$ such that α_i is a root of a degree-2 polynomial over $\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_{i-1})$, for all i.

Proof sketch:
- constructible numbers form a field
  i.e. α, β constructible $\Rightarrow \alpha-\beta, \alpha/\beta$ constructible
($\Rightarrow$) each operation solves a particular degree-2 polynomial
($\Leftarrow$) α^{2^n} constructible $\Rightarrow \sqrt{\alpha}$ constructible, + quadratic formula

Classic impossible challenges:
- angle trisection ($\cos \theta/3$) [Wantzel 1836]
- cube doubling ($\sqrt[3]{2}$) "Delian" [Descartes 1637]
- circle squaring (π) [Lindemann 1882]

Constructible regular polygons: [Gauss]
$\Leftrightarrow n = 2^k p_1 p_2 \cdots p_j$ where $p_i = 2^{u_i}+1$ is a Fermat prime
3, 5, 17, 257, 65537, ?
- excludes 7, 9, 11, 13, 14, 18, 19, 21, 22, 23, ...