Kempe's Universality Theorem:

6.885
L2.2
Sept. 13, 2004

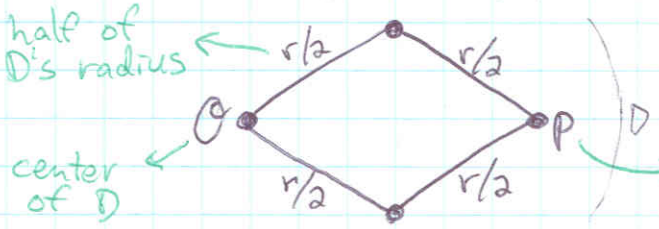
[Kempe 1876 had bug; Thurston; Kapovich & Millson 2002; King 1999]

Given any algebraic (planar) curve $\varphi(x,y) = \sum_i c_i x^{p_i} y^{q_i} = 0$
and any disk D in the plane, (closed disk)
there is a linkage and a vertex of that linkage such that the trace of the vertex is the portion of curve within D

- restriction to finite region like D is necessary in any connected linkage with a pinned vertex

Kempe's proof:

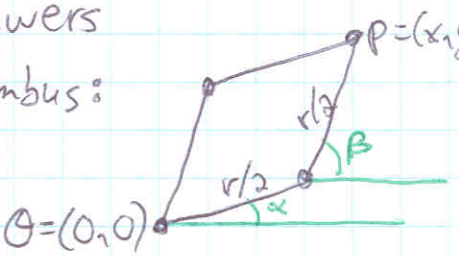
- start with rhombus to constrain point p within D:



- goal: constrain $p=(x,y)$ to satisfy $\varphi(x,y)=0$

Main trick: use trig to avoid powers

- write x & y with trig w.r.t. rhombus:



$$x = \frac{r}{2}\cos\alpha + \frac{r}{2}\cos\beta$$

$$y = \frac{r}{2}\sin\alpha + \frac{r}{2}\sin\beta = \frac{r}{2}\cos\left(\alpha - \frac{\pi}{2}\right) + \frac{r}{2}\cos\left(\alpha - \frac{\pi}{2}\right)$$

- apply trig identity $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$ to polynomial $\varphi(x,y) = \sum_i c_i x^{p_i} y^{q_i}$

- result: $\varphi(x,y) = c + \sum_i c_i \cos(r_i\alpha + s_i\beta + \delta_i)$

(c: const., c_i: const., r_i: int., s_i: int., δ_i: 0 or $\pm\pi/2$)