Algorithms for Aleksandrov's theorem

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OPEN: is there an efficient algorithm for constructing the (unique) convex polyhedron realizing a given convex polyhedral metric?

— reduces to algorithms for Cauchy's Rigidity Theorem (L11.5)

"The general solution to the problem of determining the structure of a polytope from its net, i.e., finding the actual polytope edges, is hopelessly difficult. Only for the tetrahedron can a complete answer be found."
—Aleksandrov 1958 (translated)

Sabitov's algorithm: [Sabitov 1996] (for Cauchy problem: P is triangulated)

— for each edge e:

— look at two incident triangles

— dihedral angle at e determined by distance between Δ apexes v_1, v_2

— imagine removing tetrahedron $T = e, v_1, v_2$ from polyhedron P
→ in P', replace e with edge $v_1 v_2$.

— $\underbrace{\mathrm{vol}(P)}_{\text{"known"}} = \mathrm{vol}(P') - \delta \cdot \underbrace{\mathrm{vol}(T)}_{\text{involves } \|v_1 - v_2\|}$, $\delta = \begin{cases} +1 & \text{if } v_1 v_2 \text{ internal} \\ -1 & \text{if } v_1 v_2 \text{ external} \end{cases}$

$\Rightarrow \mathrm{vol}(P') =$ polynomial in δ & $\|v_1 - v_2\|$.

— $\mathrm{vol}(P')$ also a root of a (large) polynomial

— substitute $\Rightarrow$ polynomial equation in δ & $\|v_1 - v_2\|$

— algebraic manipulation + squaring
$\Rightarrow$ polynomial equation in $\|v_1 - v_2\|$

— find all roots & guess one
$(2^{\Theta(n)})$

$\Rightarrow 2^{\Theta(n^2)}$ time, plus time to actually find roots (depends on desired accuracy)

$\rightsquigarrow$ for Aleksandrov problem, $O(n^2)$ shortest paths, $O(n^4)$ intersections
$\Rightarrow 2^{O(n^5)}$ time (plus roots)