6.885
L12.03
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Aleksandrov's Theorem: [1941]

every convex polyhedral metric on manifold homeomorphic to the sphere is realized by a unique convex polyhedron (possibly degenerating to a doubly covered plane polygon).

Proof (sketch):

- **Uniqueness:**
  - draw all shortest paths between 2 vertices
  - this "mesh" includes all edges of any polyhedral realization
  - Cauchy Rigidity theorem ⇒ unique convex realization

- **Existence:**
  - induct on n = #vertices
  - base case: $n \le 4$ (triangle or tetrahedron)
  - total curvature of all vertices $= 4\pi = 720°$ (consequence of "Gauss-Bonnet Theorem")
  - so $n \ge 5$ ⇒ two vertices x, y have curvatures $\alpha, \beta < \pi = 180°$
  - along shortest path between x & y, paste edge of doubly covered triangle
    - ↳ adds new vertex (apex of Δ)
    - ↳ adds material at x & y
  - continuously vary angles of Δ @ x & y from $\emptyset$
    - ↳ angle @ $x \to \alpha/2 (< 90°)$ ⇒ x flattens
    - ↳ angle @ $y \to \beta/2$ ⇒ y flattens

  ⇒ continuous "path" (in manifold of metrics) from original metric to metric with one less vertex ($-2 + 1 = -1$)
  (x & y ↙ ↳ apex of Δ)

  - argue continuity of realizability using Implicit Function Theorem (nonconstructive) □