Flexible polyhedra

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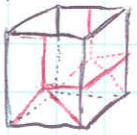
- Cauchy's Rigidity Theorem says that any convex polyhedron is rigid using edges as hinges
  (it would have to become nonconvex & can't immediately)
- also, rigid with finite number of extra creases [Connelly 1980]
  (second-order)
- contrasting example: paper bag [Balkcom, Demaine, Demaine 2004]
  - rigid with usual creases
  - can be flattened with finite additional creases



- generic (almost all) nonconvex triangulated polyhedra are infinitesimally rigid [Gluck 1976] ← necessary?
- but nonetheless, flexible nonconvex polyhedra exist:
  - Bricard "octahedron" (~1900)
    - self-intersecting (more of a linkage)

  - Connelly's polyhedron [1978]
    - modification to avoid self-intersection
    - 30 faces, 50 edges, 22 vertices
  - Steffen's polyhedron
    - simplification with 14 Δs, 9 vertices
  - need >8 vertices (⇒ 9 is minimal)

Bellow's Theorem: [Connelly, Sabitov, Walz 1997]
Volume remains constant during flex of nonconvex polyhedron

Volume polynomials: [Sabitov 1996; Astrelin & Sabitov 1999]
volume of a polyhedron is the root of a polynomial p depending only on combinatorial structure & edge/diagonal lengths (squared)
⇒ constant during continuous motion (finite # choices)

- degree must be exponential:

pop in or out

- in fact, $\leq 2^{\#\text{edges}}$ [Fedorchuk & Pak 2004]
- true even for generalized (self-intersecting) polyhedra
  ↳ volume = sum of signed volumes of tetrahedra from point