Cauchy's rigidity theorem [Cauchy 1813; Steinitz 1934]

6.885
L11.4
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If two convex polyhedra are combinatorially equivalent (same graph structure) and corresponding faces are congruent, then polyhedra are congruent

Proof by contradiction: Consider counterexample P, P'

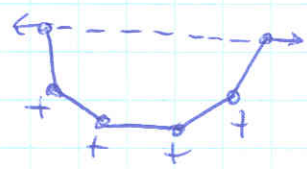
- for each vertex pair v, v':
  slice polyhedron with ε-sphere centered at vertex

  $\Rightarrow$ spherical polygon, edge lengths = face angles
  angles = dihedral angles

- faces congruent $\Rightarrow$ edge lengths match in v vs. v'
- P, P' incongruent $\Rightarrow$ angles don't match for some v, v' pair
- for such a pair v, v':
  - label vertex in v's spherical polygon + if larger angle in v
    − if smaller angle in v
    ($\emptyset$ otherwise)
  - can't be all + (or $\emptyset$) or all − (or $\emptyset$):

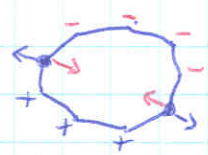
    Cauchy arm lemma: opening all angles of convex open chain increases distance between endpoints
    (plane or sphere)

    $\Rightarrow$ edge length would not be preserved.



$\Rightarrow \geq 2$ alternations: $\overset{\frown}{+}\,+ \ldots + \overset{\frown}{-}\,- \ldots - \overset{\frown}{}$

- can't be just 2, by arm lemma:



$\Rightarrow \geq 4$ alternations: $+ \overset{\frown}{-} \overset{\frown}{+} \overset{\frown}{-}$

$\Rightarrow$ # alternations $\geq 4V$ in subgraph of +/− edges.

- +/− labels extend to edges of polyhedron
- alternation at vertex $\underset{1-1}{\Longleftrightarrow}$ alternation in incident face
- $\leq 2k$ alternations in face of degree $2k$ or $2k+1$

  $\Rightarrow 4V \leq$ # alternations $\leq 2f_3 + 4f_4 + 4f_5 + 6f_6 + 6f_7 + \cdots$

- $E = \frac{1}{2}(3f_3 + 4f_4 + 5f_5 + 6f_6 + 7f_7 + \cdots)$
- $V = 2 + E - F = 2 + \frac{1}{2}(f_3 + 2f_4 + 3f_5 + 4f_6 + 5f_7 + \cdots)$

$\Rightarrow 4V = 8 + 2f_3 + 4f_4 + 6f_5 + 8f_6 + 10f_7 + \cdots$ $\longrightarrow$ ※ $\square$