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**Admin:**

In-class quiz Wed 4/13 (open notes)
(Coverage through today's lecture.)

Project presentations start Wed 4/27.
(Meet with TAs this week.)

**Today:**

- [ ] "Gap groups" & bilinear maps
- [ ] BLS (Boneh, Lynn, Shacham) signatures
- [ ] Three-way key agreement (Joux)
- [ ] Identity-based encryption (IBE) (Boneh, Franklin)

"Gap group" is one in which

- DDH is easy ("Decision Diffie Hellman")

  [Recall: given $(g, g^a, g^b, g^c)$, to decide if $ab = c \pmod{\text{order}(g)}$]

but
- CDH is hard ("Computational Diffie Hellman")

  [Recall: given $(g, g^a, g^b)$, to compute $g^{ab}$]

(Note that CDH easy $\Rightarrow$ DDH easy)

This difference in difficulty between DDH ("easy") and CDH ("hard") forms a "gap".

— How can one construct a "gap group"?

— What good would that be?

## Bilinear maps

Suppose: $G_1$ is group of prime order $q$, with generator $g$

"shadow group" → $G_2$ is group of prime order $q$, with generator $h$

[we use multiplicative notation for both groups]

See Fig. (next page) → and there exists a (bilinear) map

$$e: G_1 \times G_1 \longrightarrow G_2$$

such that

$$\boxed{(\forall a,b)\ e(g^a, g^b) = h^{ab}}$$

!!!

$e(g,g) = h$

$$= e(g, g^{ab})$$
$$= e(g,g)^{ab}$$
$$= e(g, g^b)^a$$
$$= e(g, g^a)^b$$
$$= e(g^b, g^a)$$
$$\ldots$$

Bilinear maps also called "pairing functions"

They have an enormous number of applications.*

We are, of course, interested in efficiently computable bilinear maps.

* google: "The pairing-based crypto lounge"

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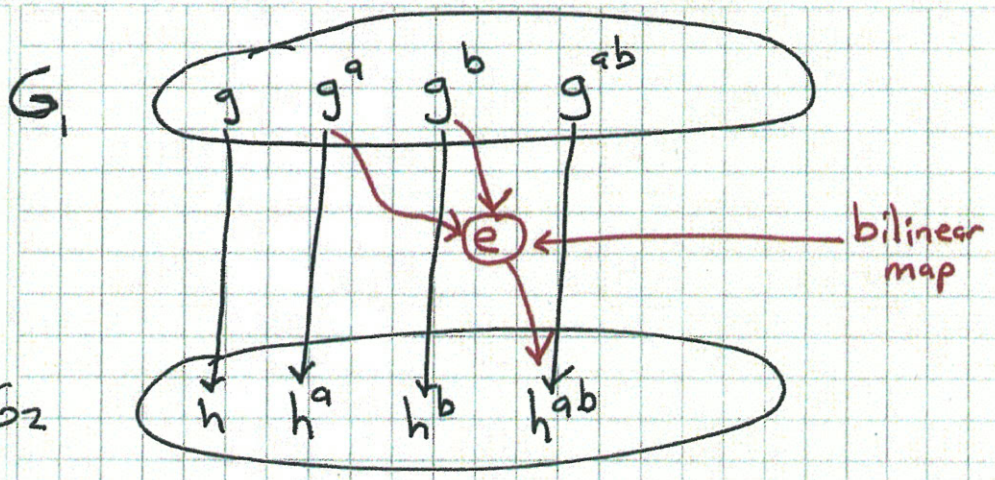


$|G_1| = |G_2| = q$ (prime)

$g$ generates $G_1$

$h$ generates $G_2$

CDH hard in $G_1$ & in $G_2$

DDH easy in $G_1$ (using $e$)

$e(g^a, g) = h^a$ computes "shadows"

**Note:** If discrete log was easy in $G_2$ then it would be easy in $G_1$.

$$DL_{G_1,g}(g^a) = DL_{G_2,h}(h^a) = a$$

**Theorem:**

If there is a bilinear map

$$e: G_1 \times G_1 \to G_2$$

between two groups of prime order $q$,

then DDH is easy in $G_1$

**Proof:**

Given $(g, g^a, g^b, g^c)$ (elements of $G_1$)

then

$$c = ab \pmod q \iff \underbrace{e(g^a, g^b) = e(g, g^c)}_{\equiv}$$

$$\underbrace{h^{ab} = h^c}_{\equiv}$$

$$ab = c \pmod q$$

So: accept $(g, g^a, g^b, g^c)$ iff $e(g^a, g^b) = e(g, g^c)$. ▨

Even though DDH is easy in $G_1$, CDH may still be hard; we may have a "gap group".

## How to construct gap groups (with bilinear maps):

- This is not simple! We give just a sketch.
- $G_1$ will be "supersinguler" elliptic curve

  e.g. elliptic curve defined by points on

  $$y^2 = x^3 + ax + b \pmod p$$

  where $p = 2 \pmod 3$, $p \geq 5$

  $a = 0$

  $b \in \mathbb{Z}_p^*$ (can choose $b = 1$)
- $G_2$ is finite field $\mathbb{F}_{p^k}$ for some small $k$

  (can use subgroups of $G_1$ & $G_2$ by choosing generators of order $\approx 2^{160}$ say... )
- $e$ (bilinear map) is implemented as a "Weil pairing" or a "Tate pairing".

**Application 1:**

# Digital signatures

(Boneh, Lynn, Shacham (2001))

Signatures are short (e.g. 160 bits)!

**Public:** groups $G_1, G_2$ of prime order $q$

pairing function $e: G_1 \times G_1 \to G_2$

$g$ = generator of $G_1$

$H$ = hash fn (C.R.) from messages to $G_1$

Note use of multiplicative notation here

**Secret key:** $x$ where $0 < x < q$

**Public key:** $y = g^x$ (in $G_1$)

**To sign message $M$:**

Let $m = H(M)$ (in $G_1$)

→ Output $\sigma = \sigma_x(M) = m^x$ (in $G_1$)

Note: Signature may be short! Just one element of $G_1$.

↑ To represent point on elliptic curve, just give $x$, then one more bit to say which $y$ is wanted (there are only two square roots of $y^2$: $\pm y$)

**To verify $(y, M, \sigma)$:**

Check $e(g, \sigma) \stackrel{?}{=} e(y, m)$ where $m = H(M)$

$e(g,m)^x$ in both cases

**Theorem:** BLS signature scheme secure against existential forgery under chosen message attack in ROM assuming CDH is hard in $G_1$.

**Application 2:**

## Three-way key agreement (Joux, generalizing DH)

**Recall DH:** $A \to B: g^a$
$B \to A: g^b$
key $= g^{ab}$

**Joux:** Suppose $G_1$ has generator $g$
Suppose $e: G_1 \times G_2$ is a bilinear map.

$A \to B, C: g^a$

$B \to A, C: g^b$

$C \to A, B: g^c$

A computes $e(g^b, g^c)^a = e(g,g)^{abc}$ ←

B computes $e(g^a, g^c)^b = e(g,g)^{abc}$ ← $= !$

C computes $e(g^a, g^b)^c = e(g,g)^{abc}$ ←

key $= e(g,g)^{abc}$

Bilinear Diffie-Hellman problem

Secure assuming "BDH" ≡

given $g, g^a, g^b, g^c, e$

hard to compute $e(g,g)^{abc}$

Four-way key agreement is open problem!

(multilinear maps!)

Application 3:

## Identity-based encryption (IBE) [Boneh, Franklin '01]

TTP (trusted third party) publishes

$G_1, G_2, e$ (bilinear map), $g$ (generator of $G_1$), $y$

where $y = g^s$ & $s$ is TTP's master secret.

Let $H_1$ be random oracle mapping names (e.g. "alice@mit.edu") to elements of $G_1^*$

Let $H_2$ be random oracle mapping $G_2$ to $\{0,1\}^*$ (PRG).

Want to enable anyone to encrypt message for Alice knowing only TTP public parameters & Alice's name

Encrypt$(y, \text{name}, M)$:

$r \xleftarrow{R} Z_q^*$ (here prime $q = |G_1| = |G_2|$)

$g_A = e(Q_A, y)$ where $Q_A = H_1(\text{name})$

output $(g^r, M \oplus H_2(g_A^r))$

**Decrypt ciphertext $c = (u, v)$:**

- Alice obtains $d_A = Q_A^s$ from TTP (once is enough)

  where $Q_A = H_1(\text{name})$.

  This is Alice's decryption key.

  Note that TTP also knows it!

  Note that message may be encrypted before Alice gets $d_A$.

- Compute $v \oplus H_2(e(d_A, u))$

$$= v \oplus H_2(e(Q_A^s, g^r))$$
$$= v \oplus H_2(e(Q_A, g)^{rs})$$
$$= v \oplus H_2(e(Q_A, g^s)^r)$$
$$= v \oplus H_2(e(Q_A, y)^r)$$
$$= v \oplus H_2(g_A^r)$$
$$= M$$

**Security:** Semantically secure in ROM assuming BDH.

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note use of additive notation

# ID-based signature (Hess 2002; Dutta survey §4.10)

master secret = $s$

master public = $P_{pub} = sP$ ($P$ generates $G_1$)

$H_1 : \{0,1\}^* \to G_1$

$H : \{0,1\}^* \times G_2 \to Z_q^*$

**Extract:** user gives ID. Public id $= H_1(ID) = Q_{ID}$

secret key $= s \cdot Q_{ID} = S_{ID}$

**Sign** $(S_{ID}, m)$: $P_1 \in_R G_1^*$

$k \in_R Z_q^*$

$r = e(P_1, P)^k$

$v = H(m, r)$

$u = v S_{ID} + k P_1$ } = signature

**Verify:** $(Q_{ID}, m, (u,v))$:

$r = e(u, P) \cdot e(Q_{ID}, -P_{pub})^v$

accept if $v = H(m, r)$

Secure against existential forgery in ROM under adaptive chosen message attacks assuming weak-DH problem is hard.

↳ given $(P, Q, sP)$ for $P, Q \in G_1$

output $sQ$