TOPIC: 6.857

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FILE UNDER:

PAGE: L14.1

Admin:

Comments on (many) project proposals on gradescope

Pset #4 out later today

Today:

Pedersen Commitments

PK encryption

El Gamal PK encryption

Semantic Security

DDH (Decision-Diffie-Hellman)

IND-CCA2 } % time

Cramer-Shoup PK enc }

Readings:

Paar: Ch 6, 7, 8

Katz: Ch 10

Pedersen Commitment Scheme

Recall: Commit(x) → "commitment to x"

Reveal(C) → "opens commitment, reveals x"

Properties: **Hiding:** Commit(x) reveals nothing about x

Binding: Can only open in one way (can't change x)

Nonmalleability (?): Can't produce commitment to e.g. x+1 from commitment to x.

values can be chosen by receiver

Setup: p, q large primes s.t. $q \mid p-1$ (e.g. p "safe prime")

g generator of order-q subgroup of $\mathbb{Z}_p^*$

(e.g. if p safe then $\langle g \rangle = Q_p$ = squares mod p)

$h = g^a$ a secret h generates $\langle g \rangle$ as well

Commit(x): $x \in \mathbb{Z}_q$ (i.e. $0 \le x < q$)

Sender chooses random $r \in \mathbb{Z}_q$

Commit(x) = $c = g^x h^r \bmod p$

Reveal: Sender reveals x and r

Receiver verifies that $c = g^x h^r \pmod p$

Pedersen commitment (cont.)

Hiding: Given $c = g^x h^r$

"Perfectly Hiding" (Adversary could have ∞ computational power...)

Can in principle be opened to any $x' \in \mathbb{Z}_q$, for some r'

$$\left.\begin{aligned} g^x h^r &= g^{x'} h^{r'} \\ g^x g^{ar} &= g^{x'} g^{ar'} \\ g^{x+ar} &= g^{x'+ar'} \end{aligned}\right\} \pmod p$$

$$x + ar = x' + ar' \pmod q$$

$$r' = (x - x')/a + r$$

↑ q is prime so a^{-1} exists; $r' \neq r$ since $x \neq x'$

Binding: If sender can reveal two ways

"Computationally Binding" (Sender can't compute a)

$$c = g^x h^r = g^{x'} h^{r'}$$

$$x + ar = x' + ar'$$

$$a = (x - x')/(r' - r)$$

↑ $r' \neq r$ & q is prime

$$= \text{discrete log of } h, \text{ base } g, \text{ mod } p$$ ∎

Non-malleable: Nope.

If $c = \text{Commit}(x) = g^x h^r$

then $c' = \text{Commit}(x) = g \cdot (g^x h^r) = g^{x+1} h^r$

(Some applications don't need non-malleability)

Public-key encryption:

Let λ = "security parameter" (i.e. "key size")

Then $1^\lambda = \underbrace{11\cdots1}_{\lambda}$ λ 1's in a row. Length = λ

Need three algorithms:

(1) $\text{Keygen}(1^\lambda) \longrightarrow (PK, SK)$

(2) $E(PK, m) \longrightarrow c$

Encryption takes $m \in$ message space M

to $c \in$ ciphertext space C

(with given public key PK)

Encryption may be <u>randomized</u>.

(3) $D(SK, c) \longrightarrow m$

Decryption is <u>deterministic</u>

s.t. (<u>Correctness condition</u>)

$$(\forall (PK, SK))(\forall m)\ D(SK, E(PK, m)) = m$$

El-Gamal PK encryption (Taher El Gamal, 1984)

Let $G = \langle g \rangle$ be a cyclic group with generator g.

(Keygen may output description of g & G, given λ.)

Keygen:

Pick x at random from $[0 \ldots |G|-1]$

Let $SK = x$.

Let $PK = g^x$

Output (PK, SK) (& description of G, if needed)

Encryption:

randomized!

Pick k at random from $[0 \ldots |G|-1]$

Assume message m represented as element of G.

Let $y = g^x$ be PK of recipient

Output $c = (g^k, m \cdot y^k)$ as ciphertext

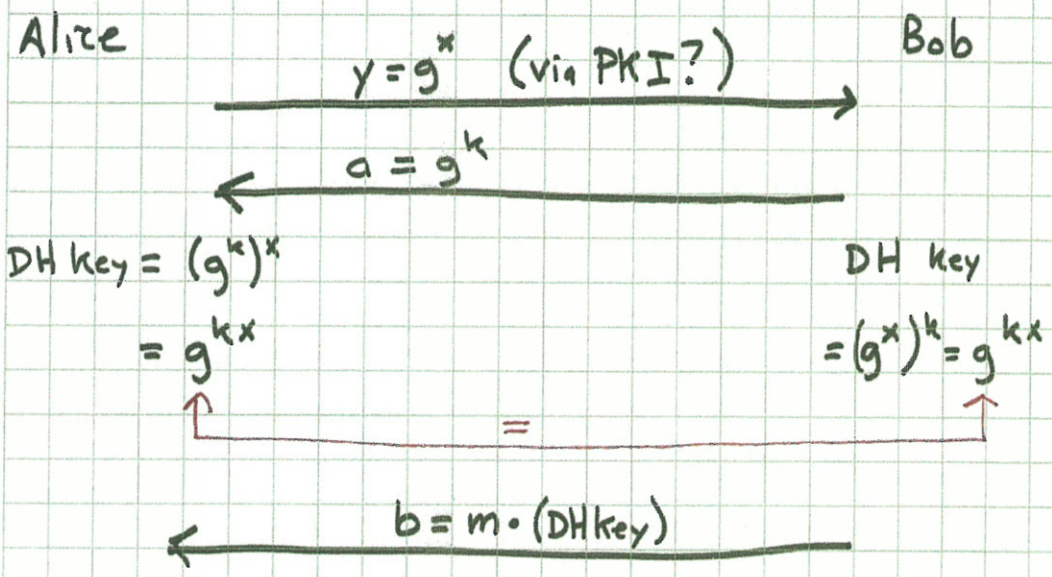
Decryption:

Let $c = (a, b)$ be received ciphertext

Let $m = b / a^x$. Output m.

[Correctness follows since $a^x = g^{kx} = g^{xk} = y^k$.]

El Gamal encryption related to DH key exchange:



Encrypt by multiplying by DH key.

Decrypt by dividing by DH key.

How to define security for PK encryption?

We'll see two definitions:

① "semantic security" (Goldwasser & Micali)

② "adaptive chosen ciphertext attack" (ACCA) secure

(≈ to IND-CCA we saw for symmetric encryption)

"Game" definition of semantic security:

Phase I ("Find"):

- Examiner generates (PK, SK) using $Keygen(1^\lambda)$
- Examiner sends PK to Adversary
- Adversary computes for polynomial (in λ) time, then outputs two messages m_0, m_1 of same length, and "state information" s. $[m_0 \neq m_1$ required$]$

Phase II ("Guess"):

- Examiner picks $b \xleftarrow{R} \{0,1\}$, computes $c^* = E(PK, m_b)$
- Examiner sends c^*, s to Adversary
- Adversary computes for polynomial (in λ) time, then outputs $\hat{b}$ (his "guess" for b).

Adversary "wins" game if $\hat{b} = b$.

Def: A PK encryption scheme is semantically secure if $\text{Prob}[\text{Adv wins}] \leq \frac{1}{2} + \text{negligible}$

Fact: In order for a PK encryption scheme to be semantically secure, it must necessarily be randomized.* (Randomized encryption is necessary but not sufficient for semantic security.)

for stateless enc →

Is El Gamal PK encryption semantically secure?

* More precisely: it can't be stateless & deterministic. It may be randomized, or stateful, or both.

DDH (Decision Diffie-Hellman Assumption):

Given a group G with generator g:

It is hard/infeasible to decide whether a given triple of elements was generated as

$$(g^a, g^b, g^c) \quad [a, b, c \text{ random}]$$

or as

$$(g^a, g^b, g^{ab}) \quad [a, b \text{ random}]$$

Recall: CDH ≡ computing g^{ab} from g^a & g^b is hard

That is, if DDH holds in a group, you can't even <u>recognize</u> the DH key g^{ab} when it is given to you! (You can't distinguish it from a random element.)

<u>Theorem:</u> DDH ⇒ CDH

<u>Proof:</u> If ¬CDH, then ¬DDH (contrapositive).

If you can compute g^{ab} from g^a and g^b (i.e. ¬CDH) then you can decide if given third element <u>is</u> g^{ab} (i.e. ¬DDH). ▨

Theorem (Tsiounis & Yung):

El Gamal is semantically secure in G

$\Longleftrightarrow$

DDH holds in G

- Semantic security may not be enough for some applications.
- El Gamal is malleable:

  Given $E(m) = (g^k, m \cdot y^k)$

  it is easy to produce $E(2m) = (g^k, (2 \cdot m) \cdot y^k)$

  without knowing m!

- More generally, El Gamal is homomorphic:

  Given $c_1 \in E(m_1) = (g^r, m_1 \cdot y^r)$

  & given $c_2 \in E(m_2) = (g^s, m_2 \cdot y^s)$

  can produce $c_1 \cdot c_2 = (g^{r+s}, (m_1 \cdot m_2) \cdot y^{r+s})$

  $\in E(m_1 \cdot m_2)$

- Product of ciphertexts yields an encryption of product of plaintexts.
- Special case: multiplying by $E(1) = (g^s, y^s)$ re-randomizes encryption.

- What is stronger notion of security for PK encryption? (e.g. one that excludes malleability...)
- "IND-CCA2 secure" (ACCA secure = secure under adaptive chosen ciphertext attack)
  ≈ IND-CCA secure defn we saw for symmetric enc.
- Similar to semantic security defn, except that Adv allowed access to decryption oracle, too.
  (He has PK so access to encryption oracle already there.)
  (As before, may <u>not</u> use oracle to decrypt challenge ciphertext during "guess" phase.)

IND-CCA2 (ACCA) security game:

Phase I ("Find"):

- Examiner generates (PK, SK) using $\text{Keygen}(1^\lambda)$
- Examiner sends PK to Adversary
- Adversary computes for polynomial (in λ) time,

new ⇒ having access to a decryption oracle $D(SK, \cdot)$

then outputs two messages m_0, m_1 of same length, and "state information" s. $[m_0 \neq m_1$ required$]$

Phase II ("Guess"):

- Examiner picks $b \xleftarrow{R} \{0,1\}$, computes $c_* = E(PK, m_b)$
- Examiner sends c_*, s to Adversary
- Adversary computes for polynomial (in λ) time,

new ⇒ having access to a decryption oracle $D(SK, \cdot)$ except on input c_*

then outputs $\hat{b}$ (his "guess" for b).

Adversary wins if $\hat{b} = b$.

Def: PK encryption method is IND-CCA2 secure (ACCA-secure) if

$$\text{Prob}[\text{Adv wins}] \leq \frac{1}{2} + \text{negligible}$$

How to make El Gamal IND-CCA2 secure?

- Cramer-Shoup method is such an extension of El Gamal.
- Let G_q be a group of prime order q

  (e.g. $G_q = Q_p$, where $p = 2q+1$, p & q prime).
- Keygen:

$$g_1, g_2 \xleftarrow{R} G_q$$

$$x_1, x_2, y_1, y_2, z \xleftarrow{R} Z_q$$

$$c = g_1^{x_1} g_2^{x_2}$$

$$d = g_1^{y_1} g_2^{y_2}$$

$$\boxed{h = g_1^{z}}$$

EG

$$PK = (g_1, g_2, c, d, h)$$

H = hash fn mapping G_q^3 to Z_q

$$SK = (x_1, x_2, y_1, y_2, z)$$

- **Enc(m) [where $m \in G_q$]:**

  $r \xleftarrow{R} Z_q$ EG

  $u_1 = g_1^r$ EG

  $u_2 = g_2^r$

  $e = h^r \cdot m$ EG

  $\alpha = H(u_1, u_2, e)$

  $v = c^r d^{r\alpha}$

  ciphertext $= (u_1, u_2, e, v)$ EG

- **Decrypt (u_1, u_2, e, v):**

  $\alpha = H(u_1, u_2, e)$

  Check: $u_1^{x_1 + y_1 \alpha} u_2^{x_2 + y_2 \alpha} \stackrel{?}{=} v$

  If not equal, reject

  else output $m = e / u_1^z$ EG

Note: $u_1^{x_1} u_2^{x_2} = g_1^{r x_1} g_2^{r x_2} = c^r$

$u_1^{y_1} u_2^{y_2} = d^r$

$u_1^z = g_1^{rz} = h^r$ EG

Theorem: Cramer-Shoup is IND-CCA2 secure (i.e. secure against adaptive chosen ciphertexts) if

1. DDH holds in G_q
2. H satisfies a certain condition
   ($\approx$ "target collision resistance")

Thus, our strongest notion of security for PK encryption is in fact achievable, albeit at some cost in terms of speed & complexity.