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Admin:

In-class quiz Wed 4/15 (open notes)

Quiz coverage through today's lecture

Friday recitation = quiz review

Some previous quizzes (& solns) are posted.

Project presentations start 4/29

Today:

- "Gap groups" & bilinear maps
- BLS (Boneh, Lynn, Shacham) signatures
- Three-way key agreement (Joux)
- Identity-based encryption (IBE) (Boneh, Franklin)

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"Gap group" is one in which

- DDH is easy ("Decision Diffie Hellman")

  [Recall: given (g, g^a, g^b, g^c), to
  decide if $ab = c \pmod{order(g)}$
  ]

but
- CDH is hard ("Computational Diffie Hellman")

  [Recall: given (g, g^a, g^b), to
  compute g^{ab}

(Note that CDH easy $\Rightarrow$ DDH easy)

This difference in difficulty between DDH ("easy") and CDH ("hard") forms a "gap".

— How can one construct a "gap group"?

— What good would that be?

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Bilinear maps

Suppose: G_1 is group of prime order q, with generator g

"shadow group" G_2 is group of prime order q, with generator h

[we use multiplicative notation for both groups]

See Fig. (next page) and there exists a (bilinear) map

$$e : G_1 \times G_1 \longrightarrow G_2$$

such that

$e(g,g) = h$

$$\boxed{(\forall a,b)\ e(g^a, g^b) = h^{ab}} \quad !!!$$

$$= e(g, g^{ab})$$

$$= e(g,g)^{ab}$$

$$= e(g, g^b)^a$$

$$= e(g, g^a)^b$$

$$= e(g^b, g^a)$$

$$\ldots$$

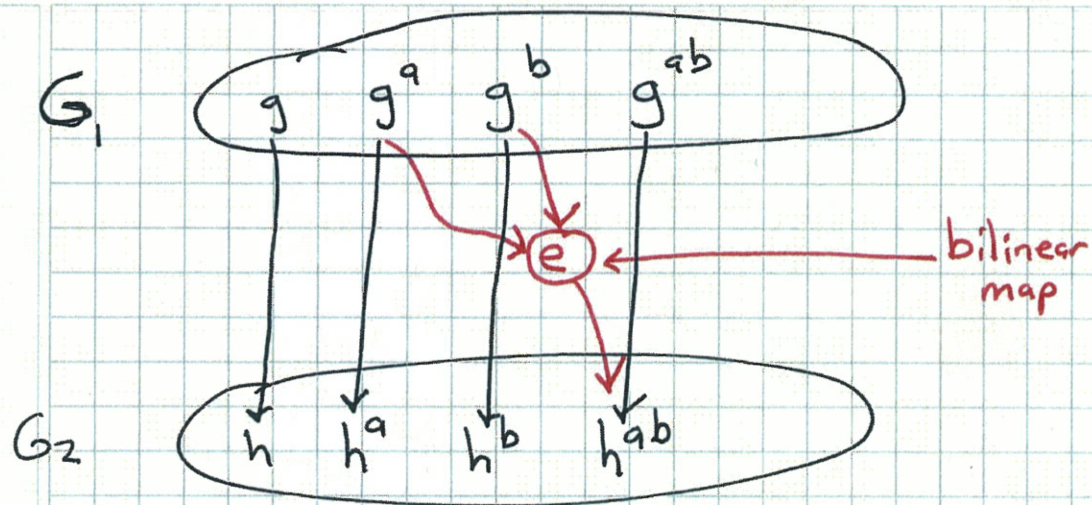
Bilinear maps also called "pairing functions"

They have an enormous number of applications.*

We are, of course, interested in efficiently computable bilinear maps.

* google: "The pairing-based crypto lounge"

"shadow group"



$|G_1| = |G_2| = q$ (prime)

$\boxed{e(g^a, g) = h^a \text{ computes "shadows"}}$

g generates G_1

h generates G_2

CDH hard in G_1 & in G_2

DDH easy in G_1 (using e)

<u>**Note:**</u> If discrete log was easy in G_2 then it would be easy in G_1.

$$DL_{G_1,g}(g^a) = DL_{G_2,h}(h^a) = a$$

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Theorem:

If there is a bilinear map

$$e : G_1 \times G_1 \to G_2$$

between two groups of prime order q,

then DDH is easy in G_1

Proof:

Given (g, g^a, g^b, g^c) (elements of G_1)

then

$$c = ab \pmod q \iff \underbrace{e(g^a, g^b) = e(g, g^c)}_{\equiv}$$

$$\underbrace{h^{ab} = h^c}_{\equiv}$$

$$ab = c \pmod q$$

So: accept (g, g^a, g^b, g^c) iff $e(g^a, g^b) = e(g, g^c)$.

$\blacksquare$

Even though DDH is easy in G_1, CDH may still be hard; we may have a "gap group".

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How to construct gap groups (with bilinear maps):

- This is not simple! We give just a sketch.
- G_1 will be "supersingular" elliptic curve

  e.g. elliptic curve defined by points on

  $$y^2 = x^3 + ax + b \pmod{p}$$

  where $p = 2 \pmod{3}$, $p \geq 5$

  $a = 0$

  $b \in \mathbb{Z}_p^*$ (can choose $b=1$)
- G_2 is finite field $\mathbb{F}_{p^k}$ for some small k

  (can use subgroups of G_1 & G_2 by choosing generators of order $\approx 2^{160}$ say...)
- e (bilinear map) is implemented as a "Weil pairing" or a "Tate pairing".

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Application 1: Digital signatures

(Boneh, Lynn, Shacham (2001))

Signatures are short (e.g. 160 bits)!

Public: groups G_1, G_2 of prime order q

pairing function $e: G_1 \times G_1 \rightarrow G_2$

g = generator of G_1

H = hash fn (C.R.) from messages to G_1

Note use of multiplicative notation here

Secret key: x where $0 < x < q$

Public key: $y = g^x$ (in G_1)

To sign message M:

Let $m = H(M)$ (in G_1)

Output $\sigma = \sigma_x(M) = m^x$ (in G_1)

Note: Signature may be short! Just one element of G_1.

To represent point on elliptic curve, just give x, then one more bit to say which y is wanted (there are only two square roots of y^2: $\pm y$)

To verify (y, M, σ):

Check $e(g,\sigma) \stackrel{?}{=} e(y,m)$ where $m = H(M)$

$e(g,m)^x$ in both cases

Theorem: BLS signature scheme secure against existential forgery under chosen message attack in ROM assuming CDH is hard in G_1.

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Application 2: Three-way key agreement (Joux, generalizing DH)

Recall DH:

$A \to B : g^a$

$B \to A : g^b$

$\text{key} = g^{ab}$

Joux: Suppose G_1 has generator g

Suppose $e : G_1 \times G_2$ is a bilinear map.

$A \to B, C : g^a$

$B \to A, C : g^b$

$C \to A, B : g^c$

A computes $e(g^b, g^c)^a = e(g,g)^{abc}$

B computes $e(g^a, g^c)^b = e(g,g)^{abc}$

C computes $e(g^a, g^b)^c = e(g,g)^{abc}$

$= !$

$\text{key} = e(g,g)^{abc}$

Bilinear Diffie-Hellman problem

Secure assuming "BDH" $\equiv$

given g, g^a, g^b, g^c, e

hard to compute $e(g,g)^{abc}$

Four-way key agreement is open problem!

(multilinear maps!)

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Application 3:

Identity-based encryption (IBE) [Boneh, Franklin '01]

TTP (trusted third party) publishes

G_1, G_2, e (bilinear map), g (generator of G_1), y

where $y = g^s$ & s is TTP's master secret.

Let H_1 be random oracle mapping names (e.g. "alice@mit.edu") to elements of G_1^*

Let H_2 be random oracle mapping G_2 to $\{0,1\}^*$ (PRG).

Want to enable anyone to encrypt message for Alice knowing only TTP public parameters & Alice's name

Encrypt$(y, name, M)$:

$r \xleftarrow{R} Z_q^*$ (here prime $q = |G_1| = |G_2|$)

$g_A = e(Q_A, y)$ where $Q_A = H_1(name)$

output $(g^r, M \oplus H_2(g_A^r))$

Decrypt ciphertext $c = (u, v)$:

- Alices obtains $d_A = Q_A^s$ from TTP (once is enough) where $Q_A = H_1(\text{name})$.
  This is Alice's decryption key.
  Note that TTP also knows it!
  Note that message may be encrypted before Alice gets d_A.
- Compute $v \oplus H_2(e(d_A, u))$

$$
\begin{aligned}
&= v \oplus H_2(e(Q_A^s, g^r)) \\
&= v \oplus H_2(e(Q_A, g)^{rs}) \\
&= v \oplus H_2(e(Q_A, g^s)^r) \\
&= v \oplus H_2(e(Q_A, y)^r) \\
&= v \oplus H_2(g_A^r) \\
&= M
\end{aligned}
$$

Security: Semantically secure in ROM assuming BDH.

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note use of additive notation

ID-based signature (Hess 2002; Dutta survey §4.10)

master secret $= s$
master public $= P_{pub} = sP$ (P generates G_1)

$H_1 : \{0,1\}^* \to G_1$

$H : \{0,1\}^* \times G_2 \to \mathbb{Z}_q^*$

Extract: user gives ID. Public id $= H_1(ID) = Q_{ID}$
secret key $= s \cdot Q_{ID} = S_{ID}$

Sign (S_{ID}, m): $P_1 \in_R G_1^*$

$k \in_R \mathbb{Z}_q^*$

$r = e(P_1, P)^k$

$\left.\begin{array}{l} v = H(m, r) \\ u = v S_{ID} + k P_1 \end{array}\right\} = \text{signature}$

Verify: $(Q_{ID}, m, (u,v))$:

$$r = e(u,P) \cdot e(Q_{ID}, -P_{pub})^v$$

accept iff $v = H(m,r)$

Secure against existential forgery in ROM under adaptive chosen message attack assuming weak-DH problem is hard.

given (P, Q, sP) for $P, Q \in G_1$
output sQ