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Quiz 4/17 open notes

Today:

Malleability of El Gamal (Homomorphic too!)

IND-CCA2

Cramer-Shoup

RSA

Making RSA IND-CCA2 secure (OAEP)

Other aspects of RSA security

Digital Signatures

Theorem (Tsiounis & Yung):

El Gamal is semantically secure in G

$\Longleftrightarrow$

DDH holds in G

- Semantic security may not be enough for some applications.
- El Gamal is malleable:

  Given $E(m) = (g^k, m \cdot y^k)$

  it is easy to produce $E(2m) = (g^k, (2 \cdot m) \cdot y^k)$

  without knowing m!

- More generally, El Gamal is homomorphic:

  Given $c_1 \in E(m_1) = (g^r, m_1 \cdot y^r)$

  & given $c_2 \in E(m_2) = (g^s, m_2 \cdot y^s)$

  can produce $c_1 \cdot c_2 = (g^{r+s}, (m_1 \cdot m_2) \cdot y^{r+s})$

  $\in E(m_1 \cdot m_2)$

- Product of ciphertexts yields an encryption of product of plaintexts.
- Special case: multiplying by $E(1) = (g^s, y^s)$ re-randomizes encryption.

- What is stronger notion of security for PK encryption? (e.g. one that excludes malleability...)
- "IND-CCA2 secure" (ACCA secure = secure under adaptive chosen ciphertext attack)
  ≈ IND-CCA secure defn we saw for symmetric enc.
- Similar to semantic security defn, except that Adv allowed access to decryption oracle, too.
  (He has PK so access to encryption oracle already there.)
  (As before, may <u>not</u> use oracle to decrypt challenge ciphertext during "guess" phase.)

IND-CCA2 (ACCA) security game:

Phase I ("Find"):

- Examiner generates (PK, SK) using $Keygen(1^\lambda)$
- Examiner sends PK to Adversary
- Adversary computes for polynomial (in λ) time,

new ⇒ having access to a decryption oracle $D(SK, \cdot)$

then outputs two messages m_0, m_1 of same length, and "state information" s. [$m_0 \neq m_1$ required]

Phase II ("Guess"):

- Examiner picks $b \xleftarrow{R} \{0,1\}$, computes $c_* = E(PK, m_b)$
- Examiner sends c_*, s to Adversary
- Adversary computes for polynomial (in λ) time,

new ⇒ having access to a decryption oracle $D(SK, \cdot)$ except on input c_*

then outputs $\hat{b}$ (his "guess" for b).

Adversary wins if $\hat{b} = b$.

Def: PK encryption method is IND-CCA2 secure (ACCA-secure) if

$$\text{Prob}[\text{Adv wins}] \leq \frac{1}{2} + \text{negligible}$$

How to make El Gamal IND-CCA2 secure?

- Cramer-Shoup method is such an extension of El Gamal.
- Let G_q be a group of prime order q
  (e.g. $G_q = Q_p$, where $p = 2q+1$, p & q prime).
- Keygen:

$$g_1, g_2 \xleftarrow{R} G_q$$

$$x_1, x_2, y_1, y_2, z \xleftarrow{R} Z_q$$

$$c = g_1^{x_1} g_2^{x_2}$$

$$d = g_1^{y_1} g_2^{y_2}$$

$$\boxed{h = g_1^{z}}$$

EG

$$PK = (g_1, g_2, c, d, h)$$

H = hash fn mapping G_q^3 to Z_q

$$SK = (x_1, x_2, y_1, y_2, z)$$

- **Enc(m) [where $m \in G_q$]:**

  $r \xleftarrow{R} Z_q$ EG

  $u_1 = g_1^r$ EG

  $u_2 = g_2^r$

  $e = h^r \cdot m$ EG

  $\alpha = H(u_1, u_2, e)$

  $v = c^r d^{r\alpha}$

  ciphertext $= (u_1, u_2, e, v)$ EG

- **Decrypt (u_1, u_2, e, v):**

  $\alpha = H(u_1, u_2, e)$

  Check: $u_1^{x_1 + y_1 \alpha} u_2^{x_2 + y_2 \alpha} \stackrel{?}{=} v$

  If not equal, reject

  else output $m = e / u_1^z$ EG

Note: $u_1^{x_1} u_2^{x_2} = g_1^{r x_1} g_2^{r x_2} = c^r$

$u_1^{y_1} u_2^{y_2} = d^r$

$u_1^z = g_1^{rz} = h^r$ EG

Theorem: Cramer-Shoup is IND-CCA2 secure (i.e. secure against adaptive chosen ciphertexts) if

① DDH holds in G_q

② H satisfies a certain condition

(≈ "target collision resistance")

Thus, our strongest notion of security for PK encryption is in fact achievable, albeit at some cost in terms of speed & complexity.

Diffie-Hellman model of PK encryption:

- Keygen $(1^\lambda) \rightarrow (PK, SK, M, C)$

  (public key, secret key, message space, ciphertext space)

  Here $|M| = |C|$.

- $E(PK, \cdot)$ is an efficiently computable one-to-one (deterministic) map from M to C

  $c = E(PK, m)$ is (unique) ciphertext for m

- $D(SK, \cdot)$ is efficiently computable inverse:

  $D(SK, c) = D(SK, E(PK, m)) = m \quad (\forall m \in M)$

- It is hard/infeasible to decrypt with knowledge of PK but without knowledge of SK.

  SK represents "trapdoor" information that enables inversion of the (otherwise one-way) function $E(PK, \cdot)$.

RSA PK encryption (Rivest, Shamir, Adleman 1977)

Keygen:

Find two large primes p, q (e.g. $\lambda = 1024$ bits each)

$n = p \cdot q$

$\varphi(n) = |Z_n^*| = (p-1)(q-1)$

$e \xleftarrow{R} Z_{\varphi(n)}^*$ [i.e. $\gcd(e, \varphi(n)) = 1$]

$d = e^{-1} \pmod{\varphi(n)}$ [e.g. Euclid's extended alg]

$PK = (n, e)$

$SK = (d, p, q)$

$M = C = Z_n$

Encrypt:

Given $m \in Z_n$ and $PK = (n, e)$:

$$c = E(PK, m) = m^e \pmod{n}$$

Decryption:

Given $c \in Z_n$ and $SK = (d, p, q)$:

$$m = D(SK, c) = c^d \pmod{n}$$

(where $n = p \cdot q$)

Note:

p & q should be large randomly chosen primes, as security of RSA depends upon inability of adversary to factor n (from PK) into p, q.

Correctness of RSA:

Lemma: (Chinese remainder theorem or CRT)

Let $n = p \cdot q$ where p, q are distinct primes.

Then $(\forall x, y \in Z_n)$

$$x = y \pmod{n} \iff x = y \pmod{p} \;\&\; x = y \pmod{q}$$

Thus it suffices to prove RSA correct mod p; the proof mod q is the same, and CRT then implies correctness mod n.

Given $e \cdot d = 1 \pmod{\varphi(n)}$ $\quad [d = e^{-1} \pmod{\varphi(n)}]$

so $e \cdot d = 1 + t \cdot (p-1)(q-1)$ for some t

and $e \cdot d = 1 \pmod{(p-1)}$ $\quad [d = e^{-1} \pmod{p-1}]$

Correctness of RSA means

$$(m^e)^d = m \pmod{n} \quad \text{for all } m \in Z_n$$

By CRT we only need to prove

$$(m^e)^d = m \pmod{p} \quad \text{for all } m \in Z_p$$

We consider two cases:

Case 1: $m = 0 \pmod p$

Trivial: $0^{ed} = 0 \pmod p$

Case 2: $m \neq 0 \pmod p$

i.e. $m \in \mathbb{Z}_p^*$

so $m^{p-1} = 1 \pmod p$ [Fermat]

Then $m^{ed} = m^{1+u\cdot(p-1)} \pmod p$

where $u = t\cdot(q-1)$

$m^{ed} = m \cdot (m^{p-1})^u \pmod p$

$= m \cdot 1^u$

$= m$

$\therefore$ $m^{ed} = m \pmod p$ for all $m \in \mathbb{Z}_p$

& $m^{ed} = m \pmod q$ for all $m \in \mathbb{Z}_q$ (similarly)

and $m^{ed} = m \pmod n$ for all $m \in \mathbb{Z}_n$ (by CRT)

Thus $(\forall m \in \mathbb{Z}_n)\ D(SK, E(PK, m)) = m$ ▨