6.851	Lecture 23	May 4, 2010

Compact suffix arrays/trees:

TODAY * Grossi & Vitter: [STOC 2000; SICOMP 2005]

$(1/\varepsilon + O(1))\underbrace{|T| \lg |\Sigma|}_{OPT}$ bits — first

$O\left(\frac{|P|}{\log_{|\Sigma|} |T|} + |\text{output}| \cdot \log^{\varepsilon}_{|\Sigma|} |T|\right)$ query

- Ferragina & Manzini: [FOCS 2000] — compressed

$5 \cdot \underbrace{H_k(T)}_{\text{kth order empirical entropy}} \cdot |T| + O\left(\frac{|T|}{\lg |T|}(|\Sigma| + \lg\lg |T|) + |T|^{\varepsilon} \cdot |\Sigma|^{O(|\Sigma|)}\right)$ bits — for <u>any</u> fixed k

$= \sum_{|w|=k} \underbrace{\Pr\{w \text{ occurring}\}}_{= \#\text{occurrences}/|T|} \cdot H_0(\text{string of successor characters of } w)$

= # bits/char. in OPT code depending on last k chars.

$O(|P| + |\text{output}| \cdot \lg^{\varepsilon} |T|)$ query

- Sadakane: [J. Alg. 2003] — large alphabets

$1/\varepsilon \cdot H_0(T) \cdot |T| + O(|T| \lg\lg |\Sigma| + |\Sigma| \lg |\Sigma|)$ bits

$O(|P| \lg |T| + |\text{output}| \cdot \lg^{\varepsilon} |T|)$ query

- Grossi, Gupta, Vitter: <u>succinct</u> [SODA 2003]

$H_k(T) \cdot |T| + O\left(|T| \lg |\Sigma| \cdot \frac{\lg\lg |T|}{\lg |T|}\right)$ bits

$O(|P| \lg |\Sigma| + \lg^{O(1)} |T|)$ query

- <u>low-space</u> construction [Hon, Sadakane, Sung – FOCS 2003; Hon, Lam, Sadakane, Sung, Yu – Alg. 2007]

$O(|T| \lg |\Sigma|)$ working

- suffix-tree ops. like maximal common substrings [Han & Sadakane – CPM 2002; Sadakane – TCS 2007]
- document retrieval [Sadakane – J. Disc. Alg. 2007]
- dynamic [Chan, Hon, Lam, Sadakane – T. Alg. 2007]

Compressed suffix arrays: [Grossi & Vitter – STOC 2000]

follow divide & conquer of linear-time suffix tree/array alg. [L7], but 2-way instead of 3-way

- start: text $T_0 = T$
  size $n_0 = n$
  suffix array $SA_0 = SA$
  $SA[i]$ = index in T of $\overbrace{i\text{th suffix}}^{\text{in sorted order of suffixes}}$ of T
- step: $T_{k+1} = \langle (T_k[2i], T_k[2i+1]) \text{ for } i = 0, 1, \ldots, n/2 \rangle$
  $n_{k+1} = n_k/2 = n/2^k$
  $SA_{k+1} = \frac{1}{2} \cdot$ extract even entries of SA_k
- represent SA_k using SA_{k+1}:
  1. $\underline{\text{even-succ}}_k(i) = \begin{cases} i & \text{if } SA_k[i] \text{ is even} \\ j & \text{if } SA_k[i] = SA_k[j] - 1 \text{ is odd} \end{cases}$
  2. $\underline{\text{even-rank}}_k(i)$ = # even suffixes preceding ith suffix
     = # even values in $SA_k[:i]$
  3. $SA_k[i] = 2 \cdot SA_{k+1}[\text{even-rank}_k(\underbrace{\text{even-succ}_k(i)}_{\text{round to even suffix} \Rightarrow \text{in } SA_{k+1}})] - (1 - \text{is-even}_k(i))$

     $\underbrace{\underbrace{\underbrace{\text{even-rank}_k(\text{even-succ}_k(i))}_{\text{name of even suffix in } SA_{k+1}}}_{\text{index of even suffix in } T_{k+1}}}_{\text{index of even suffix in } T_k}$ $\quad$ unround
- stop recursion at level $\ell = \lg\lg n \Rightarrow n_\ell = n/\lg n$
  $\Rightarrow$ can afford naive $n_\ell \lg n_\ell \le n$-bit ptr. encoding

$\Rightarrow O(\lg\lg n)$ query to $SA[i]$, given $O(1)$-time is-even-suffix & even-succ & even-rank

$\underline{\text{is-even-suffix}_k}(i) = \begin{cases} 1 & \text{if } SA_k[i] \text{ is even} \\ 0 & \text{else} \end{cases}$ $\rightarrow n_k$ bits

$\underline{\text{even-rank}_k} = \text{rank}_1$ in is-even-suffix $\rightarrow O\left(n_k \frac{\lg\lg n_k}{\lg n_k}\right)$ bits

$\underline{\text{even-succ}_k}$:

- trivial for $SA_k[i]$ even ($n_k/2$ such i's)
- store answers for $SA_k[i]$ odd ($n_k/2$ such i's)
- order by i $\Rightarrow$ $\text{even-succ}_k(i) = \text{odd-answers}[\underbrace{\text{odd-rank}_k(i)}_{= i - \text{even-rank}_k(i)}]$

$= \text{order by odd suffix } T_k[SA_k[i]:]$

$= \text{order by } (T_k[SA[i]], T_k[\underbrace{SA_k[i]+1}_{\text{even}}:])$

$= \text{order by } (T_k[SA[i]], T_k[SA_k[\text{even-succ}_k(i)]:])$ (def. of even-succ)

$= \text{order by } (T_k[SA[i]], \underbrace{\text{even-succ}_k(i)}_{\text{answer!}})$ (SA_k is a suffix array)

- actually store these pairs, in order by value
- assume $|\Sigma| = 2$ here

$\Rightarrow$ storing sorted array of $n_k/2$ values, $2^k + \lg n_k$ bits each

- store leading $\lg n_k$ bits of each value v_i via unary differential encoding: $0^{\text{lead}(v_1)}10^{\text{lead}(v_2)-\text{lead}(v_1)}1\ldots$

  $\Rightarrow n_k/2$ 1's & $\le 2^{\lg n_k} = n_k$ 0's $\Rightarrow \le \frac{3}{2} n_k$ bits

  - $\text{lead}(v_i) = \text{rank}_0(\text{select}_1(i)) - \text{rank}_0(\text{select}_1(i-1))$
- store trailing 2^k bits of each v_i explicitly in array

  $\Rightarrow 2^k \cdot n_k/2 = n/2$ bits

$\Rightarrow \frac{1}{2}n + \frac{3}{2}n_k + O\left(\frac{n_k}{\lg\lg n_k}\right)$ bits

Total: $\sum_{k=0}^{\lg\lg n} \left(n_k + \frac{1}{2}n + \frac{3}{2}n_k + O\left(\frac{n_k}{\lg\lg n_k}\right)\right)$

$= \frac{1}{2} n \lg\lg n + 5n + O\left(\frac{n}{\lg\lg n}\right)$ bits

TOO BIG

Compact suffix array: $O(n)$ bits

- store only $1/\varepsilon + 1$ levels of recursion:
  $0, \varepsilon\ell, 2\varepsilon\ell, \ldots, \ell = \lg\lg n$
- represent $SA_{k\varepsilon\ell}$ using $SA_{(k+1)\varepsilon\ell}$ similarly:
  - "even" → "divisible by $2^{\varepsilon\ell}$" = "in $SA_{(k+1)\varepsilon\ell}$"
  - is-even$_k$ = $n_{k\varepsilon\ell}$-bit vector as before + rank$_1$ struct.
  - succ$_k(i) = j$ where $SA_{k\varepsilon\ell}[i] = SA_{k\varepsilon\ell}[j] - 1$
    stored in $n + 2n_{k\varepsilon\ell} + O\left(\frac{n_{k\varepsilon\ell}}{\lg\lg n_{k\varepsilon\ell}}\right)$ bits like even-succ
- to compute $SA_{k\varepsilon\ell}[i]$:
  1. follow succ$_k$ repeatedly until j in $SA_{(k+1)\varepsilon\ell}$
  2. recurse: $SA_{(k+1)\varepsilon\ell}[\text{rank}_1(j)]$ → is-even$_k$ bit vector
  3. multiply by $2^{\varepsilon\ell}$, subtract by # steps in 1
- $\leq 2^{\varepsilon\ell} = \lg^{\varepsilon} n$ succ$_k$ calls per level

$\Rightarrow O(\lg^{\varepsilon} n \lg\lg n) = O(\lg^{\varepsilon'} n)$ query to $SA[i]$

- space: $\sum_{k=0}^{1/\varepsilon} \left(n_{k\varepsilon\ell} + n + 2n_{k\varepsilon\ell} + O\left(\frac{n_{k\varepsilon\ell}}{\lg\lg n_{k\varepsilon\ell}}\right)\right)$
  $= (1/\varepsilon + 6)n + O\left(\frac{n}{\lg\lg n}\right)$ bits
- optimizations:
  - succ$_k$ free at level $k=0 \Rightarrow \Sigma \leq 4n_{\varepsilon\ell} = O\left(\frac{n}{\lg^{\varepsilon} n}\right)$
  - store is-even$_k$ as succinct dictionary (with rank)
    $\Rightarrow \lg\binom{n_{k\varepsilon\ell}}{n_{(k+1)\varepsilon\ell}} + o \sim n_{(k+1)\varepsilon\ell} \lg \underbrace{\frac{n_{k\varepsilon\ell}}{n_{(k+1)\varepsilon\ell}}}_{2^{\varepsilon\ell}} + o = O\left(n \frac{\lg\lg n}{\lg^{\varepsilon} n}\right)$

$\Rightarrow 1/\varepsilon \cdot n + O\left(\frac{n}{\lg\lg n}\right)$ bits

$\boxed{\text{OPEN}}$: $O(n)$ bits & $o(\lg^{\varepsilon} n)$ query

Compact suffix tree given suffix array:

[Munro, Raman, Rao – J. Alg. 2001]

- store compressed binary trie on $2n+1$ nodes in $4n + o(n)$ bits using balanced parens. [L22]
  - ↪ "skipping the skips": can't store edge lengths
- during search for P, maintain letter depth of node:
  to descend $x \to y$:
  - compute length of edge by finding longest match between leftmost-leaf(y) & rightmost-leaf(y), starting at letter depth of x (leaf rank + SA)
  - check that P matches these letters

$\Rightarrow O((|P| + |output|) \cdot \text{cost for SA lookup})$

↪ # matches via # leaves in subtree

↪ enumerate k matches via leaf select + rank

Grossi & Vitter achieve $O\left(\frac{|P|}{\log_{|\Sigma|} |T|} + |output| \cdot \log^{\varepsilon}_{|\Sigma|} |T|\right)$

← word RAM

Leaf ops:

leaf = ( ))

leaf-rank = $\text{rank}_{()}$(here)

leaf-select = $\text{select}_{()}(i)$

leaves in subtree = $\text{rank}_{()}$(matching) of parent) − $\text{rank}_{()}$(here)

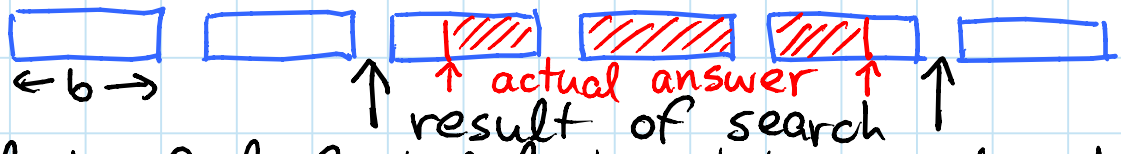
leftmost-leaf = leaf-select(leaf-rank(here) + 1)

rightmost-leaf = leaf-select(leaf-rank(matching) of parent − 1))

"Succinct" suffix tree, given suffix array

[Munro, Raman, Rao: fixed]

- use suffix tree above on every bth suffix in suffix order (keep every bth leaf in tree)
- search in tree narrows to interval of size-b blocks in SA:



$\Rightarrow$ need to find first & last match in a block

- lookup table:
  for any b b-bit strings in sorted order & any $\le b$-bit query string, find first & last prefix match

  2^{b^2}
  2^b
  $O(\lg b)$

  $\Rightarrow O(2^{b^2+b} \lg b)$ bits $= O(\sqrt{n})$ if $b \le \frac{1}{2}\sqrt{\lg n}$
- to search in a block:
  0. locate the b suffixes in T using SA — $O(b)+$
  1. read next b bits from P & from all b suffixes (in T) — $(O(1)+O(b))$
  2. repeat $\lceil |P|/b \rceil$ times — $\cdot O(P/b)$

$\Rightarrow O(|P| + \underbrace{b}_{\text{forgotten in [MRR]}})$ time $= O(|P| + \lg\lg n)$ if $b = O(\lg\lg n)$

$\times$ cost of SA query $\Rightarrow O(|P| \lg^{\varepsilon} n)$

- $O(\frac{n}{b})$ bits plus size of suffix array