6.851	Lecture 13	Mar. 18, 2010

Tree decompositions:

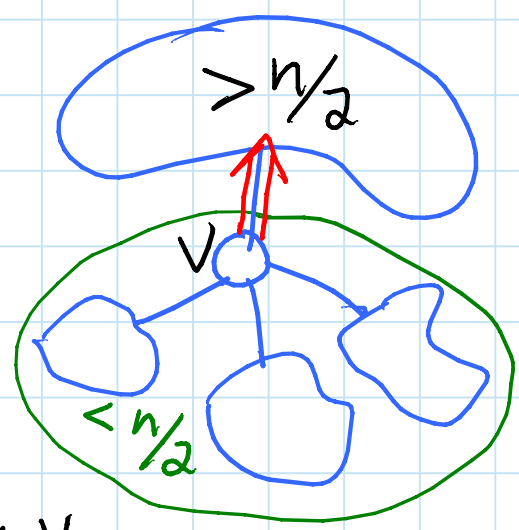
- preferred paths — Tango trees [L2] & link-cut trees [L16]
- heavy-light decomp. — link-cut trees [L16] & many other apps
- separator decomp. — tree search [TODAY]
- ART decomp. / leaf trimming — level ancestors [TODAY]

Separator theorem on trees: [Jordan 1864 – "Sur les assemblages de lignes" – J. Reine Angew. Math.]

Any tree on n vertices has a vertex whose removal disconnects the tree into components of size $\leq n/2$

Proof:
- pick any vertex v
- if not done, (exactly) one component of $T-v$ has size $> n/2$
- walk one step into that component
- new component containing v has size $< n/2$

$\Rightarrow$ never go back to v $\Rightarrow$ terminate $\square$



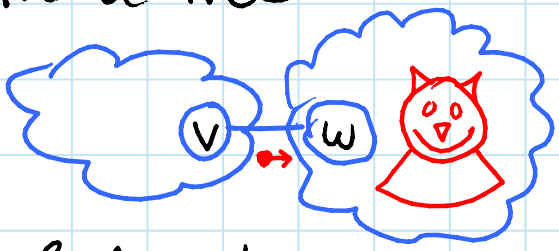
Separator decomposition:

- apply separator theorem → root of new tree
- recurse on components → children subtrees

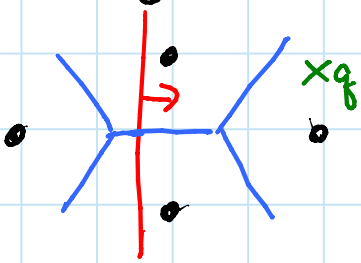
⇒ depth of new tree $= O(\lg n)$

Tree search: [Ben-Asher, Farchi, Newman – SICOMP 1999]

- looking for a cat (node) in a tree
- given an edge (v, w), oracle tells you which subtree (v or w's) has cat



- separator decomp. lets you find cat in $O(\Delta \lg n)$ oracle calls for max. degree Δ
  - $O(\Delta \frac{\lg n}{\lg \Delta})$ possible [Laber & Nogueira – ENDM 2001]
- best tree can be found in $O(n)$ time [Moses, Onak, Weimann – SODA 2008]
- applications:
  - file system synchronization
  - bug detection in tree of modules
  - search in a dynamic Voronoi diagram of points in convex position [Aronov, Bose, Demaine, Gudmundsson, Iacono, Langerman, Smid – LATIN 2006]



ART decomposition: [Alstrup, Husfeldt, Rauhe – FOCS 1997]

- define bottom tree rooted at each maximally high node with $\leq \lg n$ leaves below
  $\Rightarrow$ bottom trees are disjoint
  & each has compressed size $O(\lg n)$
  (similar trick in LCAs [Lecture 8/9])
- top tree on remaining nodes
  $\Rightarrow$ has $\leq n/\lg n$ leaves
  (charge a top leaf to $> \lg n$ leaves in bottom trees below)
- recurse in top tree
  $\Rightarrow \lg n / \lg\lg n$ recursive levels

Marked ancestor problem: [ART 1997]

- given rooted tree ~ here, static
- each node can be marked or unmarked
- updates: mark(v) & unmark(v)
- query: lowest marked ancestor of v

Bounds:

- $O\left(\frac{\lg n}{\lg \lg n}\right)$ query, $O(\lg \lg n)$ update [TODAY]
- $\Omega\left(\frac{\lg n}{\lg \lg n}\right)$ query assuming $\lg^{O(1)} n$ update
  - "chronogram technique" [Spring '05, L17]
- $\Omega(\lg \lg n)$ update, even in a path: colored predecessor problem with $u = n$

Application: dynamic method dispatching in OOP

- tree = inheritance among classes
- mark = class implements method x
- query = call to x

$\boxed{\text{OPEN}}$: DAGs, for multiple inheritance

Marked ancestor upper bound: recursive ART decomposition

Query: each bottom tree:

- has $\le \lg n \le w$ leaves $\Rightarrow <w$ branches
- maintain bit vector of which compressed edges (nonbranching paths) have a marked node
  - partially ordered by ancestry (e.g. pre-order)
- each node stores bitmask of its ancestor compressed edges

$\Rightarrow$ mask & least significant 1 bit
EITHER locates desired compressed edge
OR says must be in top tree in $O(1)$ time

- maintain van Emde Boas DS $(u=n)$ on each compressed edge

$\Rightarrow$ in bottom case, $O(\lg \lg n)$ to finish query

- in top case, recurse on top tree with parent of root of bottom tree

$\Rightarrow O\left(\frac{\lg n}{\lg \lg n} + \cancel{\lg \lg n}\right)$ query

Update:

- each node stores which bottom tree it's in
- predecessor update for compressed edge + bit vector update for bottom tree

$\Rightarrow O(\lg \lg n)$

Decremental connectivity in a tree: DYNAMIC GRAPHS

updates = edge deletions

query = is there a path $v \to w$? OR what's root of v's component?

= marked ancestor problem with mark only

- assume all $n-1$ edges eventually deleted

① $O(\lg n)$ amortized update, $O(1)$ query
(also possible with link-cut/Euler tour trees [L16])
- each node stores connected component i.d.
- delete(v,w):
  - run DFS from both v & w in parallel
  - stop when one DFS stops $\Rightarrow$ smaller comp.
  - update all nodes in smaller component to new i.d.
- component containing an updated node shrinks by $2\times$

$\Rightarrow O(\lg n)$ updates to any node
(similar trick in union-find DS)
- can also store i.d. $\to$ root of component mapping

② $O(1)$ amortized for a path
- split into chunks of length $\lg n$
- store each chunk as a bit vector
- use ① to store which chunks have a cut
  $\Rightarrow O(n/\lg n)$ updates cost $O(\lg n)$
- query: find right chunk in $O(1)$ via ①
  shift & least sig. 1 bit within a chunk

③ $O(1)$ amortized for top tree
- use ① on $O(n/\lg n)$ compressed paths
- use ② on each nonbranching path

④ $O(1)$ amortized for bottom trees

as above
- maintain bit vector of which $\underbrace{\text{compressed paths}}_{\leq \lg n}$ have a cut
  - again partially ordered by depth
- preprocess mask for ancestors of each node
- query within compressed path = mask, LSB
- use ② on each nonbranching path

⑤ nonrecursive ART decomposition with ③ & ④