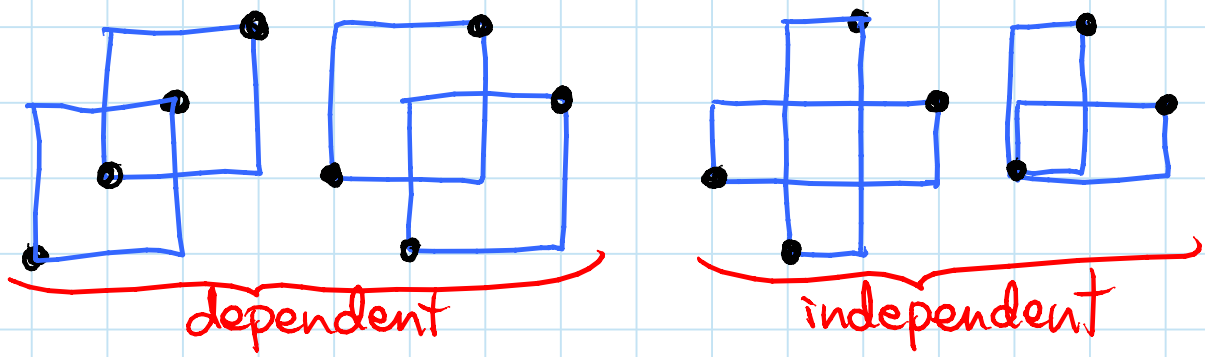
6.851	Lecture 2	Feb. 4, 2010

Lower bounds: [Demaine, Harmon, Iacono, Kane, Pătraşcu]

Independent rectangles are unsatisfied & no corner is strictly inside another



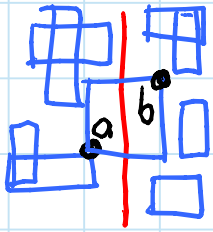
Theorem: $OPT \geq |\text{input}| + \frac{1}{2}$ max # independent rectangles

Signed rectangles:  types

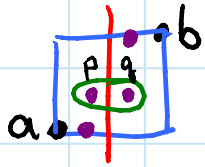
- ⊠-satisfied if all  rectangles have another pt.
- $OPT_{⊠}$ = smallest ⊠-satisfied superset of points

Lemma: $OPT_{⊠} \geq |\text{input}| +$ max # independent ⊠-rectangles

Proof: ① find rectangle in indep. set & vertical line hitting just it



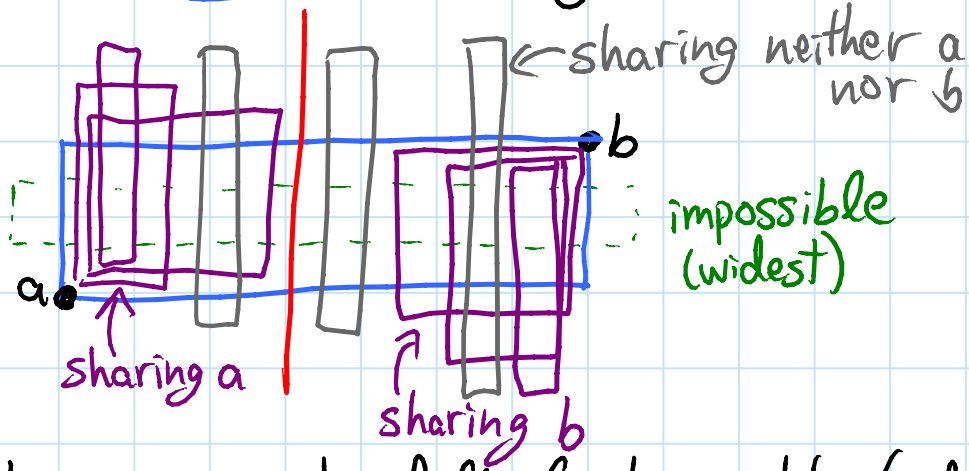
② find horizontally adjacent pts. of $OPT_{⊠}$ in rect. crossing line



③ charge indep. rectangle to those points

Assume input x & y coords. all distinct

① : take the widest rectangle

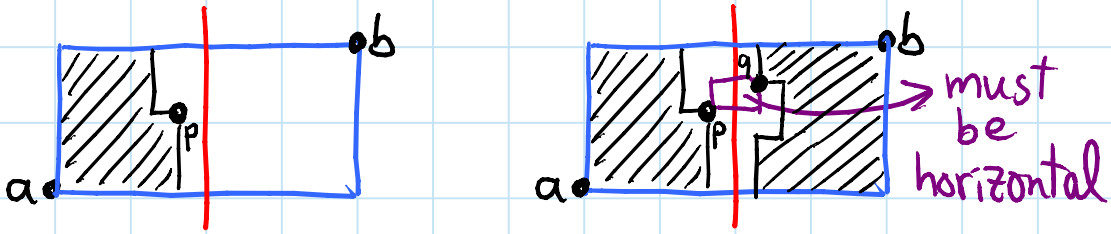


- sharing-a rects. left of sharing-b's (indep.)
- sharing-neithers fit in between vertical edges

⇒ room left for vertical line

② : take p = topmost rightmost point in rectangle & left of line (e.g. a)

q = bottommost leftmost point in rectangle & right of line & not below p (e.g. b)



③ : p & q are not in any other common rectangle

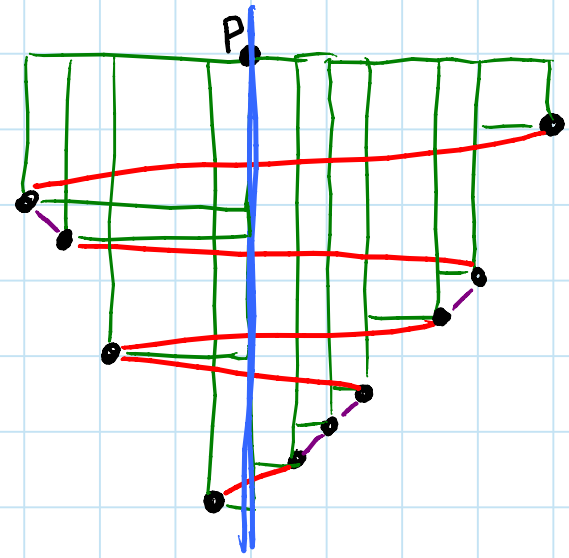
⇒ won't get charged again

- at most one is in input; other is added

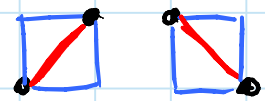
□

Wilber's second lower bound: [Wilber - SICOMP 1989]

- for each point p:
  - look at orthogonally visible points below p
  - count # alternations between left/right of p
- sum over all p



Proof: independent rectangle ∀ alternation:



$\boxed{\text{Conjecture:}}$ OPT = Θ(Wilber 2)

Key-independent optimality: [Iacono - ISAAC 2002]

- suppose key values are "meaningless"

⇒ might as well permute them uniformly at random

- claim: E[OPT] = working-set bound

  ⇒ splay trees, Greedy are key-indep. optimal
- proof sketch: $E[\text{Wilber } 2(x_i)] = \Theta(\lg t_i)$

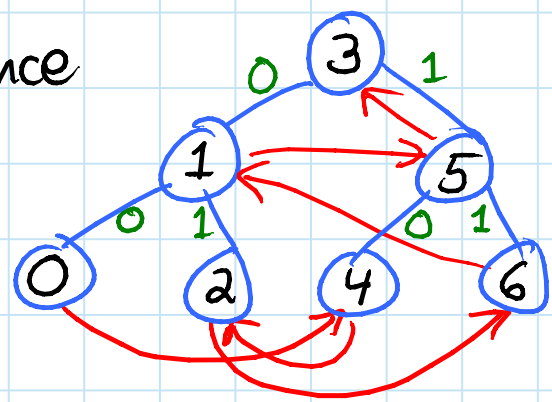
  (expected # changes to max. in random permutation)

Wilber's first lower bound: [Wilber – SICOMP 1989]

- fix a lower-bound tree P on same keys
  (e.g. perfect binary tree)
- for each node y of P:
  count # alternations in $x_1 \wedge x_2 \wedge \cdots \wedge x_n$
  between accesses in left & right subtrees of y
  (ignoring accesses to y or outside y's subtree)
- sum over all y

Example: bit-reversal sequence

000	0
001	4
010	2
011	6
100	1
101	5
110	3
111	7

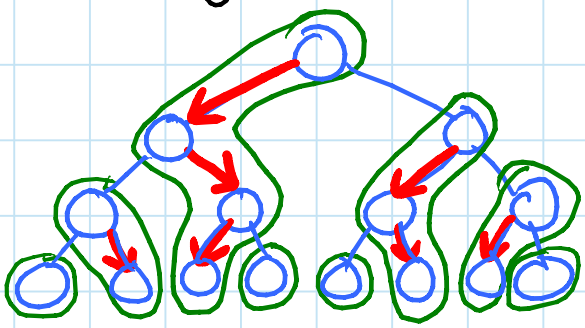


$\Rightarrow$ # alternations at y = size of y's subtree
$\Rightarrow$ Wilber 1 $= \Theta(n \lg n)$
$\Rightarrow$ OPT $= \Theta(n \lg n)$

$\boxed{\text{OPEN}}$: $\forall$ access sequence $\exists$ tree P such that
OPT $= \Theta(\text{Wilber } 1)$

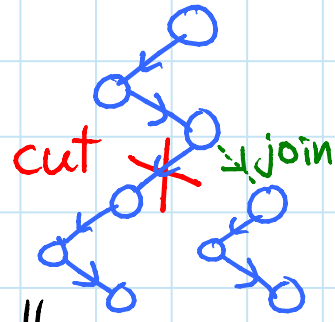
Tango trees: [Demaine, Harmon, Iacono, Pătraşcu – SICOMP 2007]

- $O(\lg \lg n)$-competitive online BST
- define preferred child of node y in P to be
  - left if accessed left subtree of y more recently
  - right if accessed right subtree of y more recently
  - none if no access to either subtree yet
- preferred path = chain of preferred child pointers
  - partition of nodes of P

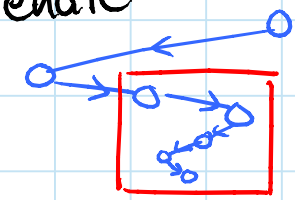


- idea: store each preferred path in auxiliary tree
  - conceptually separate balanced BST (e.g. AVL)
  - leaves link to roots of aux. trees of children paths
  - has $\le \lg n$ nodes (height of perfect P)
  - $\Rightarrow$ supports search in $O(\lg \lg n)$ time
- search starts at top aux. tree (containing root of P)
  - each jump to next aux. tree = nonpreferred edge = preferred edge change = +1 in Wilber 1
  - k jumps $\Rightarrow$ LB k, UB $(k+1) \cdot O(\lg \lg n)$
  - $\Rightarrow O(\lg \lg n)$-competitive ... if we can update preferred edges OK

Auxiliary trees:



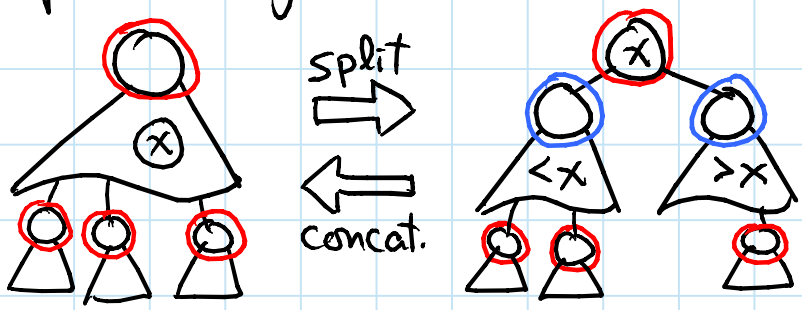
- changing a preferred child = cutting one path & joining two paths:
- if aux. trees were sorted by depth, this would be like split & concatenate
- depth $>d$ translates to interval of keys



$\Rightarrow$ can implement cuts & joins with $O(1)$ splits & concatenates
- each costs $O(\lg |\text{aux. tree}|) = O(\lg \lg n)$

In one tree: mark roots of aux. trees

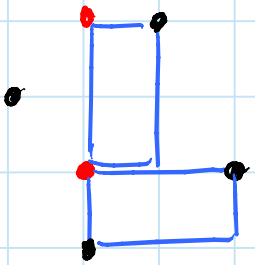
- modify split & concat. to ignore children trees & manipulate adjacent trees:



Signed Greedy:

- sweep as in Greedy
- only satisfy  boxes
- for every added point, get independent ⊠-rectangle

$\Rightarrow$ get lower bound : ⊠-Greedy



Theorem: max{⊠-Greedy, ⊠-Greedy}
$= \Theta$(biggest independent-rectangle LB)

Proof: define $OPT_{\boxtimes}$ = smallest union of ⊠-satisfying superset & ⊠-satisfying superset

$$
\begin{aligned}
(OPT \geq)\ OPT_{\boxtimes} &\\
&\geq |\text{input}| + \tfrac{1}{2}\text{ max. \# independent rectangles} \\
&\geq \tfrac{1}{2}\max\{\text{⊠-Greedy}, \text{⊠-Greedy}\} \\
&\geq \tfrac{1}{2}\max\{OPT_{\boxtimes}, OPT_{\boxtimes}\} \\
&\geq \tfrac{1}{4}(OPT_{\boxtimes} + OPT_{\boxtimes}) \\
&\geq \tfrac{1}{4} OPT_{\boxtimes}
\end{aligned}
$$

what we actually proved on p.1

$\Rightarrow$ constant-factor sandwich $\square$

Summary: So close!

Greedy	vs.	Signed Greedy
⊠ & ⊠		⊠ + ⊠
UB		LB

PROJECT: compare UBs & LBs for many pt. sets